Some Properties and Applications of Topp Leone Kumaraswamy Lomax Distribution

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Abstract

Many Statisticians have developed and proposed new distributions by extending the existing distributions. The distributions are extended by adding one or more parameters to the baseline distributions to make it more flexible in fitting different kinds of data. In this study, a new four-parameter lifetime distribution called the Topp Leone Kumaraswamy Lomax distribution was introduced by using a family of distributions which has been proposed in the literature. Some mathematical properties of the distribution such as the moments, moment generating function, quantile function, survival, hazard, reversed hazard and odds functions were presented. The estimation of the parameters by maximum likelihood method was discussed. Three real life data sets representing the failure times of the air conditioning system of an air plane, the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients and Alumina (Al2O3) data were used to show the fit and flexibility of the new distribution over some lifetime distributions in literature. The results showed that the new distribution fits better in the three datasets considered.

Keywords: Bladder cancer patients, exponentiated Lomax, maximum likelihood estimation, reversed hazard rate function, Topp Leone Kumaraswamy-G

1. Introduction

Researchers have been working on developing new probability distributions by adding parameter or parameters to the standard distributions to generalize them and make them more flexible in modeling real life data sets. The addition of one or more parameters to the standard models has made the generalized models gain wide applications in areas such as Insurance, Finance, Medicine, Environmental Sciences, etc. The Lomax (Pareto type II) distribution is a continuous probability distribution with a heavy tail having one shape and one scale parameters. The distribution was introduced by Lomax (1954) to analyse business failure data. Researchers have proposed different extensions of the Lomax distribution such as the extended Lomax by Lemonte and Cordeiro, (2013), gamma-Lomax by Cordeiro et al., (2013), exponentiated Lomax by El-Bassiouny et al., (2015), The Transmuted Weibull Lomax Distribution by Ahmed et al., (2015), Weibull Lomax by Tahir et al., (2015), Gompertz Lomax by Oguntunde et al., (2017).

The two parameter Lomax distribution has the cumulative distribution function (cdf) and probability density function (pdf) given respectively as:

$$H(x;\beta,\sigma)=1-(1+\beta x)^{-\sigma} \tag{1}$$

$$h(x;\beta,\sigma)=\beta\sigma(1+\beta x)^{-(\sigma+1)} \tag{2}$$

for $x\geq 0,\beta,\sigma>0$ where β is the scale parameter and σ is the shape parameter.

For some decades now, researchers have resorted to developing families of distributions and these families of distributions are proposed mainly to develop compound distributions. These compound probability distributions are expected to be more flexible and also fit data better than existing ones. Some of these generated families of distributions are: Topp Leone exponentiated-G by Ibrahim et al., (2020a), Topp Leone Kumaraswamy-G by Ibrahim et al., (2020b), The Kumaraswamy-G by (cordeiro and decastro, 2011), Topp Leone-G by Al-Shomrani et al., (2016), Odd Lindley-G family by Gomes-Silva et al., (2017), Gompertz-G family by Alizadeh et al., (2017), Odd Frechet G family by Haq and Elgarhy (2018), Power Lindley G family by Amal et al., (2017), Power Lindley-G Family of distributions by Hassan and Nassr (2019), Modi family of continuous probability distributions by Modi et al., (2020), etc. In this context, we proposed a generalization of the Lomax distribution based on Ibrahim et al., (2020b) which stems from the following general construction: if H denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F(x;\alpha,\lambda,\theta,\xi)=\left\{1-\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda}\right\}^{\theta} \tag{3}$$

The pdf corresponding to (3) is

$$f(x;\alpha,\lambda,\theta,\xi)=2\alpha\lambda\theta h(x;\xi)H(x;\xi)^{\alpha-1}\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda-1}\left\{1-\left[1-H(x;\xi)^{\alpha}\right]^{2\lambda}\right\}^{\theta-1} \tag{4}$$

where $H(x;\xi)$ is the cdf of the baseline distribution with parameter vector ξ.

For $x\geq 0,\alpha,\lambda,\theta,\xi\geq 0$, where equations (3) and (4) are the cdf and pdf of the TLK-G family of distributions. This paper proposes a new distribution that generalizes the Lomax distribution using the family of distribution proposed by Ibrahim et al., (2020). This is to improve the flexibility of the Lomax distribution to fit a variety of data including unimodal and bimodal shapes from different disciplines. The rest of the paper is outlined as follows. In section 2, we define the Topp Leone Kumaraswamy Lomax distribution, obtain some of its mathematical properties and distribution of order statistic. Estimation of the parameters using the method of Maximum likelihood estimation is performed in section 3. In section 4, real-life application of the distribution to data sets is provided. Section 5 presents concluding remarks. Acknowledgement is presented in section 6. Competing interest is presented in section 7 and finally, references are presented in section 8.

2. The Topp Leone Kumaraswamy Lomax (TLKLx) distribution

In this section, we derive a new continuous distribution called TLKLx distribution that generalize and extend the Lomax distribution. The cdf of the new distribution is obtained by inserting (1) into (3) to get

$$F(x;\alpha,\theta,\beta,\lambda,\sigma) = \left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta} \tag{5}$$

$$f(x;\alpha,\theta,\beta,\lambda,\sigma) = \frac{2\alpha\theta\lambda\sigma\beta}{(1+\beta x)^{(\sigma+1)}}\left[1-(1+\beta x)^{-\sigma}\right]^{\alpha-1}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda-1}\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta-1} \tag{6}$$

for $x \geq 0, \alpha, \beta, \theta, \sigma, \lambda > 0$, where α, θ, λ are the shape parameters and β is the scale parameter. The plots of pdf of the TLKLx distribution showing different shapes with different parameter values are presented in figure 1.

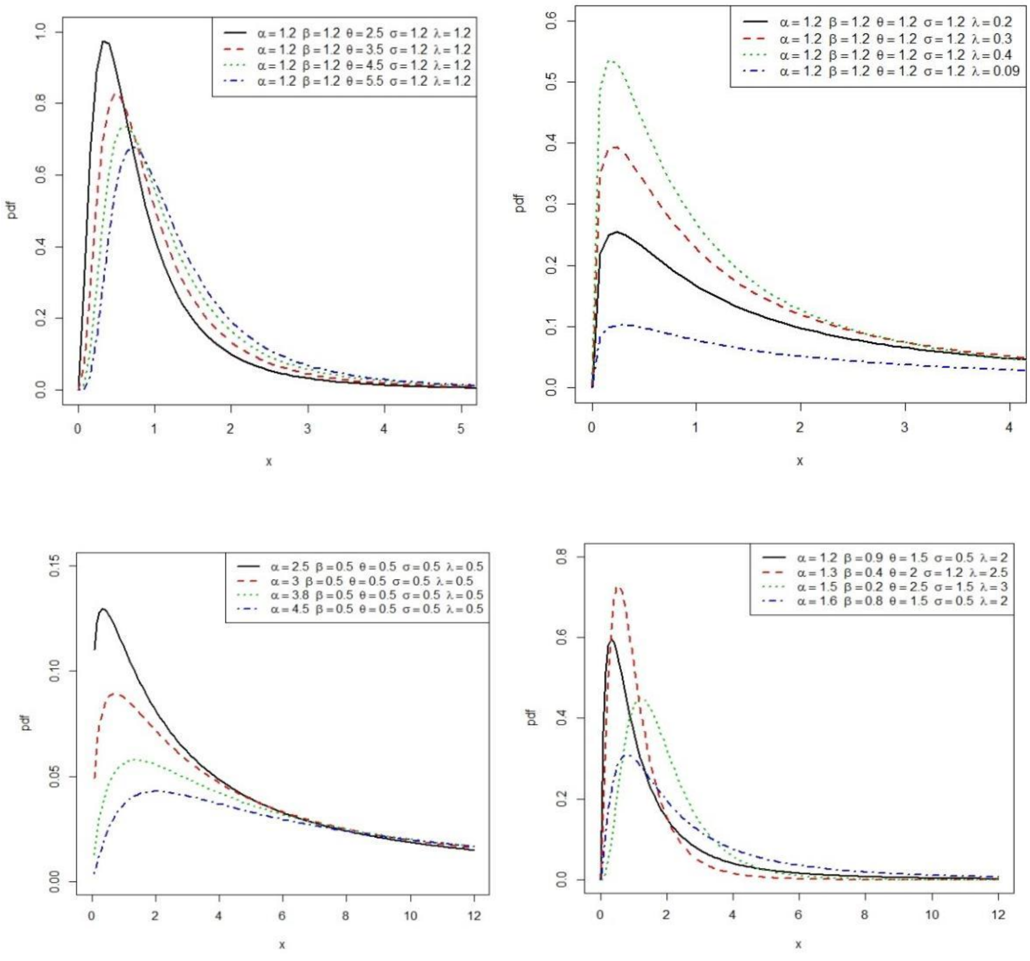

Figure 1: Plots of pdf of the TLKLx distribution for different parameter values.

2.1 Properties of the TLKLx distribution

This sub-section discusses the properties of the new models.

2.1.1 Moments

It is always pertinent to keep in mind the importance of moments in any statistical analysis particularly in applied fields. For example, through moments the important characteristics such as tendency, dispersion, skewness, and kurtosis of a distribution can be studied.

Assume Y is a Lomax distributed random variable with parameters β and σ, then the r^{th} moment of Y is given as

$$E(Y^r) = \left(\frac{\sigma}{\beta^r}\right) B(r+1, \sigma - r) \tag{7}$$

Let X be a random variable having the TLKLx distribution. Using the expansion in (7), it is easy to obtain the r^{th} moment of X as

$$E(X^r) = 2\alpha\theta\lambda\sigma\beta Q \left(\frac{\sigma(i+1)}{\beta^r}\right) B(r+1, \sigma(i+1) - r) \tag{8}$$

where $Q = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\lambda(i+1))\Gamma(\alpha(j+1))}{i!\,j!\,k!\,\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)\Gamma(\alpha(j+1)-k)}$.

2.1.2 Moment generating function

The moment generating function (mgf) is another most prominent approach to study the behavior of probability distributions. A general expression for the MGF is given as

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)dx \tag{9}$$

Since the series expansion for e^{tx} is given as

$$e^{tx} = \sum_{m=0}^{\infty} \frac{(tx)^m}{m!} \tag{10}$$

$$M_x(t) = 2\alpha\theta\lambda\sigma\beta Z \left(\frac{\sigma(i+1)}{\beta^r}\right) B(m+1, \sigma(i+1) - m) \tag{11}$$

where $Z = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\theta)\Gamma(2\lambda(i+1))\Gamma(\alpha(j+1))t^m}{i!\,j!\,k!\,m!\,\Gamma(\theta-i)\Gamma(2\lambda(i+1)-j)\Gamma(\alpha(j+1)-k)}$.

Substituting $r = 1,2,3,4$ in (8), we obtain the $\text{mean} = \mu_1'$, $\text{variance} = \mu_2' - \mu_1'^2$, $\text{skewness} = \frac{\mu_3'}{\mu_1'^{\frac{3}{2}}}$ and $\text{kurtosis} = \frac{\mu_4'}{\mu_1'^2}$.

2.1.3 Reliability

The reliability function is also known as survival function, which is the probability of an item not failing prior to some time. It can be defined as

$$R(x;\alpha,\theta,\beta,\lambda,\sigma) = 1 - F(x;\alpha,\theta,\beta,\lambda,\sigma) \tag{12}$$

The survival function $R(x)$ of TLKLx distribution is given as

$$R(x;\alpha,\theta,\beta,\lambda,\sigma) = 1 - \left\{1 - \left[1 - (1 - (1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta} \tag{13}$$

2.1.4 Hazard rate function

The hazard rate function (hrf) is defined as

$$\tau(x;\alpha,\theta,\beta,\lambda,\sigma) = \frac{f(x;\alpha,\theta,\beta,\lambda,\sigma)}{R(x;\alpha,\theta,\beta,\lambda,\sigma)} \tag{14}$$

Then, the hrf $\tau(x)$ of the TLKLx distribution is obtained as

$$\tau(x;\alpha,\theta,\beta,\lambda,\sigma) = \frac{2\alpha\theta\lambda\sigma\beta(1+\beta x)^{-(\sigma+1)}\left[1-(1+\beta x)^{-\sigma}\right]^{\alpha-1}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda-1}\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta-1}}{1-\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta}} \tag{15}$$

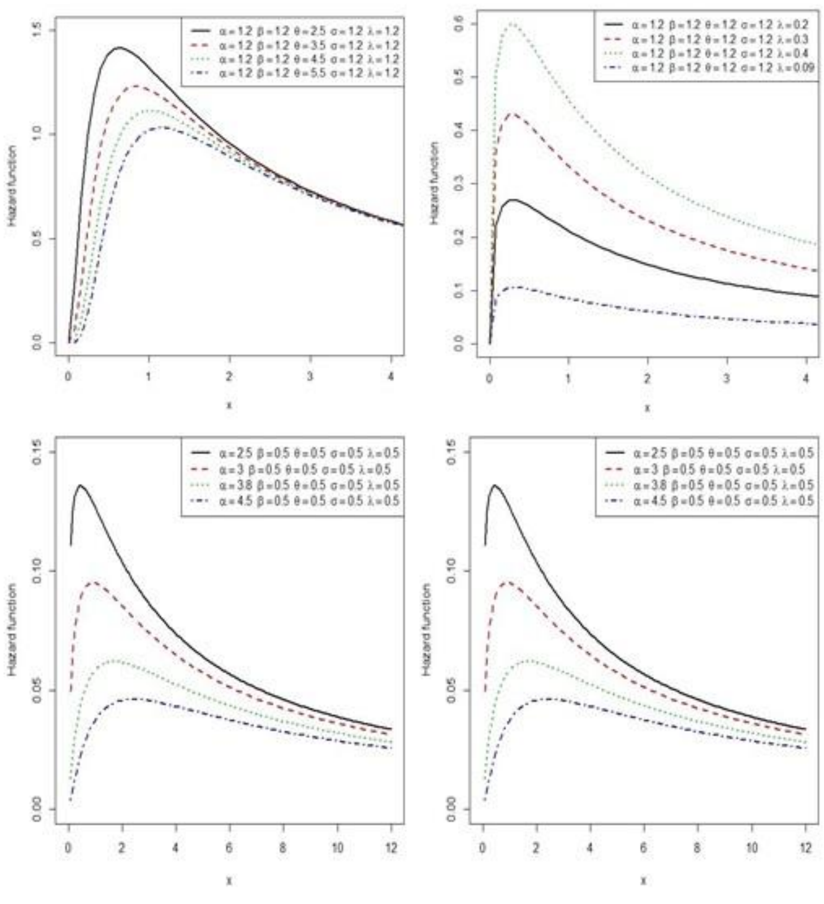

Figure 2. Plots of the hazard function of the TLKLx distribution for different parameter values.

Figure 1 and 2 show the shapes of the new model. It can be seen from the plots that the new model has different shapes like increasing, decreasing, skewed, J-shape, reversed J-shape and bathtub which makes it more flexible in fitting different kinds of real life data.

2.1.5 Reversed hazard rate function

The reversed hazard rate function for the TLKLx distribution is obtained by using the relation

$$r(x;\alpha,\theta,\beta,\lambda,\sigma)=\frac{f(x;\alpha,\theta,\beta,\lambda,\sigma)}{F(x;\alpha,\theta,\beta,\lambda,\sigma)} \tag{16}$$

$$r(x;\alpha,\theta,\beta,\lambda,\sigma)=\frac{2\alpha\theta\lambda\sigma\beta(1+\beta x)^{-(\sigma+1)}\left[1-(1+\beta x)^{-\sigma}\right]^{\alpha-1}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda-1}\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta-1}}{\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta}} \tag{17}$$

2.1.6 Odds function

The odds function for the TLExLx distribution is obtained as

$$Q(x;\alpha,\theta,\beta,\lambda,\sigma)=\frac{F(x;\alpha,\theta,\beta,\lambda,\sigma)}{R(x;\alpha,\theta,\beta,\lambda,\sigma)} \tag{18}$$

$$Q(x;\alpha,\theta,\beta,\lambda,\sigma)=\frac{\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta}}{1-\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta}} \tag{19}$$

2.1.7 Quantile function

The quantile function is defined as the inverse of the cdf and it is given as $Q(u)=F^{-1}(u)$. Using the cdf of TLKLx distribution in (5), we have

$$F(x;\alpha,\theta,\beta,\lambda,\sigma)=\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta}=u$$

$$u^{\frac{1}{\theta}}=1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}$$

$$1-u^{\frac{1}{\theta}}=\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}$$

$$\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}=1-(1-(1+\beta x)^{-\sigma})^{\alpha}$$

$$1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}=(1-(1+\beta x)^{-\sigma})^{\alpha}$$

$$\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}=1-(1+\beta x)^{-\sigma}$$

$$1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}=(1+\beta x)^{-\sigma}$$

$$\left[1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{-\sigma}}=1+\beta x$$

$$\left[1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{-\sigma}}-1=\beta x$$

$$x=\frac{1}{\beta}\left\{\left[1-\left[1-\left[1-u^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{-\sigma}}-1\right\} \tag{20}$$

Using the equation (20), we can generate random sample from the TLKLx distribution by using U as uniform random number. The median of the TLKLx distribution can be derived by substituting $u=0.5$ in (20) as follows:

$$median=\frac{1}{\beta}\left\{\left[1-\left[1-\left[1-0.5^{\frac{1}{\theta}}\right]^{\frac{1}{2\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{-\sigma}}-1\right\} \tag{21}$$

2.2 Order statistic

Let $X_1, X_2, ..., X_n$ be n independent random variable from the TLKLx distributions and let $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ be their corresponding order statistic. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r=1,2,3,...n$ denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. The pdf of the r^{th} order statistics of $X_{r:n}$ is given as

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{\infty}\frac{(-1)^i\left|(n-r+1)\right.}{i!\left|(n-r+1-i)\right.}[F(x)]^{r+i-1}f(x) \tag{22}$$

Using the cdf and pdf of TLKLx distribution in (5) and (6), we have

$$f_{r:n}(x)=\frac{2\alpha\beta\theta\lambda\sigma}{B(r,n-r+1)}\sum_{i=0}^{\infty}\frac{(-1)^i\left|(n-r+1)\right.}{i!\left|(n-r+1-i)\right.}\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta(r+i)-1}(1+\beta x)^{-(\sigma+1)}\left[1-(1+\beta x)^{-\sigma}\right]^{\alpha-1}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda-1} \tag{23}$$

Equation (23) is the r^{th} order statistic of TLKLx distribution. To obtain the minimum order statistic, we set $r=1$ in (23) to get

$$f_{r:n}(x)=\frac{2\alpha\beta\theta\lambda\sigma}{B(1,n)}\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(n)}{i!\Gamma(n-i)}\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta(r+i)-1}(1+\beta x)^{-(\sigma+1)}\left[1-(1+\beta x)^{-\sigma}\right]^{\alpha-1}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda-1} \tag{24}$$

Using binomial expansion on

$$\left\{1-\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda}\right\}^{\theta(r+i)-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\theta(1+i))}{j!\Gamma(\theta(1+i)-j)}\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda j} \tag{25}$$

$$\left[1-(1-(1+\beta x)^{-\sigma})^{\alpha}\right]^{2\lambda(j+1)-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(2\lambda(j+1))}{k!\Gamma(2\lambda(j+1)-k)}(1-(1+\beta x)^{-\sigma})^{\alpha k} \tag{26}$$

$$(1-(1+\beta x)^{-\sigma})^{\alpha(k+1)-1}=\sum_{l=0}^{\infty}\frac{(-1)^{l}\Gamma(\alpha(k+1))}{l!\Gamma(\alpha(k+1)-l)}(1+\beta x)^{-\sigma l} \tag{27}$$

Therefore, the minimum order statistics of TLKLx distribution is given as

$$f_{1:n}(x)=2n\alpha\beta\theta\lambda\sigma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma n\,\Gamma(\theta(i+1))\Gamma(2\lambda(j+1))\Gamma(\alpha(k+1))}{i!j!k!l!\Gamma n-i\,\Gamma(\theta(i+1)-j)\Gamma(2\lambda(j+1)-k)\Gamma(\alpha(k+1)-l)}(1+\beta x)^{-\sigma(l+1)-1} \tag{28}$$

Also, the maximum order statistics of TLKLx distribution is given as

$$f_{n:n}(x)=2n\alpha\beta\theta\lambda\sigma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+j+k+l}\Gamma 1\,\Gamma(\theta(i+n))\Gamma(2\lambda(j+1))\Gamma(\alpha(k+1))}{i!j!k!l!\Gamma(1-i)\Gamma(\theta(i+n)-j)\Gamma(2\lambda(j+1)-k)\Gamma(\alpha(k+1)-l)}(1+\beta x)^{-\sigma(l+1)-1} \tag{29}$$

3. Estimation

In this section, we estimate the parameters of the TLKLx distribution using maximum likelihood estimation (MLE). For a random sample, $X_1,X_2,...,X_n$ of size n from the TLKLx $(\alpha,\beta,\theta,\lambda,\sigma)$, the log-likelihood function $L(\alpha,\beta,\theta,\lambda,\sigma)$ of (6) is given as

$$\begin{aligned}\log L&=n\log 2+n\log\alpha+n\log\sigma+n\log\lambda+n\log\theta-(\sigma+1)\sum_{i=1}^{n}\log[1+\beta x_i]+(\alpha-1)\sum_{i=1}^{n}[1-(1+\beta x_i)^{-\sigma}]\\&+(2\lambda-1)\sum_{i=1}^{n}\log[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]+(\theta-1)\sum_{i=1}^{n}\log[1-(1-(1-(1+\beta x_i)^{-\sigma})^{\alpha})^{2\lambda}]\end{aligned} \tag{30}$$

Differentiating the log-likelihood with respect to $\alpha,\beta,\theta,\lambda,\sigma$ and setting the result equals to zero, we have

$$\begin{aligned}\frac{\partial\log L}{\partial\alpha}&=\frac{n}{\alpha}+\sum_{i=1}^{n}[1-(1+\beta x_i)^{-\sigma}]+(2\lambda-1)\sum_{i=1}^{n}\frac{[1-(1+\beta x_i)^{-\sigma}]^{\alpha}\log[1-(1+\beta x_i)^{-\sigma}]}{1-[1-(1+\beta x_i)^{-\sigma}]^{\alpha}}\\&+(\theta-1)\sum_{i=1}^{n}\frac{2\lambda\alpha[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda-1}[1-(1+\beta x_i)^{-\sigma}]^{\alpha}\log[1-(1+\beta x_i)^{-\sigma}]^{\alpha}}{1-[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}}=0\end{aligned} \tag{31}$$

$$\frac{\partial \log L}{\partial \beta}=\frac{n}{\beta}-(\sigma+1)\sum_{i=1}^{n}\frac{x_i}{(1+\beta x_i)}+(\alpha-1)\sum_{i=1}^{n}\frac{\sigma x_i(1+\beta x_i)^{-\sigma-1}}{[1-(1+\beta x_i)^{-\sigma}]}+(2\lambda-1)\sum_{i=1}^{n}\frac{\alpha[1-(1+\beta x_i)^{-\sigma}]^{\alpha-1}\sigma x_i(1+\beta x_i)^{-\sigma-1}}{[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{2\lambda[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda-1}\alpha[1-(1+\beta x_i)^{-\sigma}]^{\alpha-1}\sigma x_i(1+\beta x_i)^{-\sigma-1}}{[1-[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}]}=0 \quad (32)$$

$$\frac{\partial \log L}{\partial \sigma}=\frac{n}{\sigma}-\sum_{i=1}^{n}\log[1+\beta x_i]+(\alpha-1)\sum_{i=1}^{n}\frac{(1+\beta x_i)^{-\sigma}\log(1+\beta x_i)^{-\sigma}}{[1-(1+\beta x_i)^{-\sigma}]}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{2\lambda\alpha[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda-1}[1-(1+\beta x_i)^{-\sigma}]^{\alpha-1}(1+\beta x_i)^{-\sigma}\log(1+\beta x_i)^{-\sigma}}{[1-[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}]} \quad (33)$$
$$+(2\lambda-1)\sum_{i=1}^{n}\frac{\alpha[1-(1+\beta x_i)^{-\sigma}]^{\alpha-1}(1+\beta x_i)^{-\sigma}\log(1+\beta x_i)^{-\sigma}}{[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]}=0$$

$$\frac{\partial \log L}{\partial \lambda}=\frac{n}{\lambda}+2\sum_{i=1}^{n}\log[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]+(\theta-1)\sum_{i=1}^{n}\left[\frac{[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}\log[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]}{1-[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}}\right]=0 \quad (34)$$

$$\frac{\partial \log L}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log[1-[1-(1-(1+\beta x_i)^{-\sigma})^{\alpha}]^{2\lambda}]=0 \quad (35)$$

Now, equations (31), (32), (33), (34) and (35) do not have a simple form and are therefore intractable. As a result, we have to resort to non-linear estimation of the parameters using iterative procedures.

4. Application to real-life data

The first data set represents the failure times of the air conditioning system of an airplane. The data set was given by Linhart and Zucchini (1986) and it has also been used by Shanker et al., (2015). It has thirty (30) observations as follows: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

The second data set was given by Lee and Wang (2003) and it represents the remission times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients. The data set is given as:

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69.

The third data is Alumina (Al2O3) taken from the website: http://www.ceramics.nist.gov/srd/summary/ftmain.htm. This data set can also be found in Nadarajah and Kotz (2008) and the data set is given as:

5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4,5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5

The competing models used in this analysis are Kumaraswamy Exponentiated Lomax (KExLx) distribution with five parameters, Exponentiated Lomax (ExLx) distribution with four parameters and Lomax (Lx) distribution with two parameters.

- Kumaraswamy Exponentiated Lomax (KExLx)distribution (El-Batal and Kareem 2014)
  $$f(x)=\alpha\theta\beta\lambda(1+\beta x)^{-(\sigma+1)}[1-(1+\beta x)^{-\sigma}]^{\alpha\lambda-1}[1-[1-(1+\beta x)^{-\sigma}]^{\alpha\lambda}]^{\theta-1}$$
- ExLx distribution (Salem, 2014)
  $$f(x)=\theta\sigma\beta(1+\beta x)^{-(\sigma+1)}[1-(1+\beta x)^{-\sigma}]^{\theta-1}$$
- Lx distribution
  $$\sigma\beta(1+\beta x)^{-(\sigma+1)}$$

Table 1. The MLEs and Information Criteria of the models based on data set 1.

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\lambda}$	$-l$	AIC	BIC	CAIC
TLKLx	0.2743	0.0004	7.3247	2.6739	2.3638	151.3074	312.6148	319.6208	315.1148
KExLx	1.4565	0.0126	0.4519	5.1476	0.8470	151.7755	313.5511	320.5571	316.0511
ExLx	-	0.5889	5.5254	0.7859	-	154.2919	314.5837	318.7873	315.5068
Lx	-	0.0587	-	0.7014	-	155.3782	314.7563	317.5587	315.2008

Table 2. The MLEs and Information Criteria of the models based on data set 2.

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\lambda}$	$-l$	AIC	BIC	CAIC
TLKLx	3.431	0.131	0.463	0.899	2.463	410.065	830.129	844.389	830.622
KExLx	0.491	0.107	0.3482	5.244	3.430	415.842	841.685	855.945	842.176
ExLx	-	6.823	15.133	0.871	-	439.873	885.746	894.302	885.939
Lx	-	0.009	-	13.439	-	413.834	831.667	837.371	831.763

Table 3. The MLEs and Information Criteria of the models based on data set 3.

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\lambda}$	$-l$	AIC	BIC	CAIC
TLKLx	20.754	0.024	0.399	13.832	38.567	169.016	348.032	361.927	348.563
KExLx	36.909	0.594	40.656	1.454	0.6312	175.241	360.482	374.378	361.013
ExLx	-	21.767	14.765	0.062	-	194.824	395.649	403.986	395.857
Lx	-	0.0585	-	4.099	-	306.111	616.222	621.780	616.326

The tables 1-3 present the estimate of each parameter and goodness of fit for the models considered. The goodness of fits considered are the Akaike Information Criteria (AIC), Bayessian Information Criteria (BIC) and Corrected Akaike Information Criteria (CAIC). The smaller the AIC, BIC and CAIC values the better the model.

The figures 3-5 present the shapes, fit and flexibility of the new model in relation to the data sets considered. The black line represents the new model, the red line represents the KExLx, the green line represents the ExLx while the blue line represents the Lx distributions. It can be seen from the plots the the black line which represents the TLKLx distribution fits better in the three data sets considered.

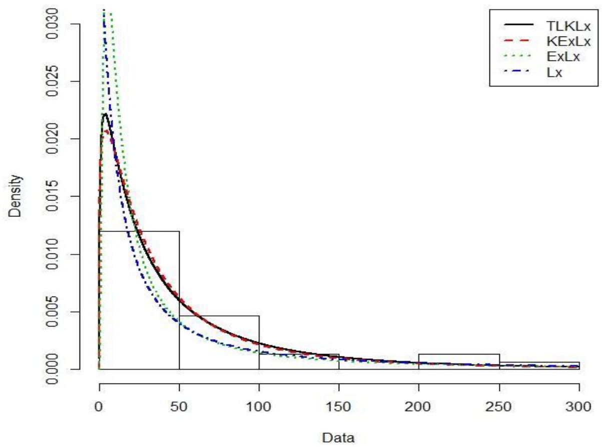

Figure 3: Fitted pdfs of the TLKLx, KExLx, ExLx and Lx models for data set 1.

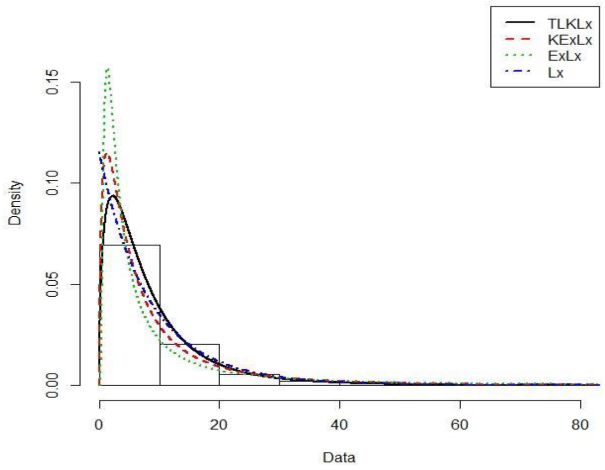

Figure 4: Fitted pdfs of the TLKLx, KExLx, ExLx and Lx models for data set 2.

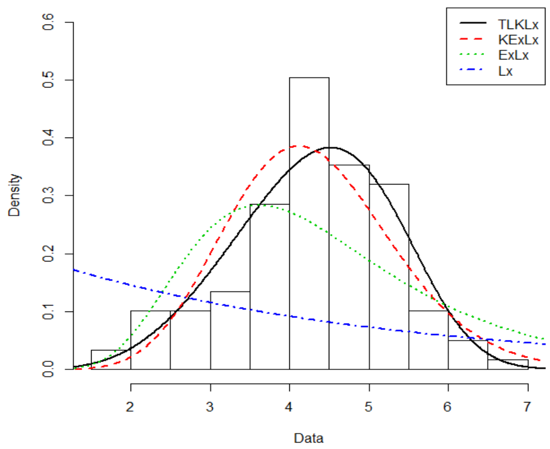

Figure 5: Fitted pdfs of the TLKLx, KExLx, ExLx and Lx models for data set 3.

5. Conclusion

This paper has developed a new distribution called the Topp Leone Kumaraswamy Lomax distribution that generalizes the Lomax distribution. Some properties of the new distribution were derived such as the survival function, hazard rate function, quantile function, the median and order statistics. The shapes of the proposed distribution were shown by plotting the graphs of the pdf and hazard function. The estimation of the model parameters by the method of the maximum likelihood was carried out using a package in *R* known as *AdequacyModel*. Application of the Topp Leone Kumaraswamy Lomax distribution to three real datasets shown from Table 1, Table 2 and Table 3 that the Topp Leone Kumaraswamy Lomax distribution is quite effective and superior in fitting the three data sets considered. Also, from the fitted pdf plots for the three data sets, it can be deduced that the new models fit the three data sets better than the competing models considered.

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7. Competing interests

The authors declare that they have no competing interests.

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