JOURNAL OF STATISTICAL MODELING & ANALYTICS (JOSMA)
(ISSN: 2180-3102)
UNIVERSITI MALAYA

Parameter Estimation for Circular Simultaneous Functional Relationship Model (CSFRM) for Unequal Variances

Fatin Najihah Badarisam[1], Mohd Syazwan Mohamad Anuar[2*] & Adzhar Rambli[3]

[1]*Faculty of Defence Science & Technology, National Defence University of Malaysia, Kem Sungai Besi, 57000 Kuala Lumpur*
[2]*Centre for Defence Foundation Studies, National Defence University of Malaysia, Kem Sungai Besi, 57000 Kuala Lumpur*
[3]*Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia*

[*]*Corresponding author: m.syazwan@upnm.edu.my*

RESEARCH ARTICLE

Received 26 July 2023
Accepted 11 September 2023
Published 25 October 2023

Abstract

In this study, we propose an extended model of the Circular Simultaneous Functional Relationship Model from the Circular Functional Relationship Model. In this case, the circular model and the circular variable will be applied assuming the error variances are not equal. All estimations of the parameter followed von Mises distribution. The angular and slope parameters are obtained using the *ms* function, while concentration parameter estimation is obtained from the *polyroot* function provided in the SPLUS statistical package. The simulation study has been conducted to assess the efficiency of the proposed model. The simulation results showed that as sample size and concentration parameters increase, all parameters' estimates are close to the true value and have a smaller bias. The illustration of real wind and wave direction data from two different bases of data is given to show its practical applicability. We note that the proposed method for parameter estimation works well with the proposed model in the case of unequal error variances as it provides good estimates.

Keywords: circular simultaneous functional relationship model, *ms* function, parameter estimates, *polyroot* function, unequal variances

1. Introduction

Circular data is located at the circumference of the circle and can be called directional data (Badarisam et al. 2020). It is usually measured in degrees or radians. Circular data have been applied in various scientific fields such as medicine, meteorology, geology, earth sciences and astronomy. The application of medicine has been discussed by Jammaladaka & Sengupta (2001) in the recovery of orthopaedic patients that can be accessed by measuring the angle of knee flexion (Jammaladaka et al. 1986). Meteorologists used circular data to study the wind and wave directions (Johnson & Wehrly, 1977; Hussin et al., 2004; Gatto & Jammalamadaka, 2007). While the directional data is used by geologists to model cross-bedding data (Jones & James 1969) and earthquake displacement direction (Rivest, 1997).

Next, in earth sciences is to measure the ocean waves (Caires and Wyatt, 2003). Ahmad et al. (2020) recently applied circular data to study a new crescent moon visibility criterion.

The circle's most common continuous probability distribution is a von Mises distribution (Mardia & Jupp 2000). It is analogous to the normal distribution because it has some similar characteristics to the normal distribution (Hassan et al. 2012). Von Mises (1918, as cited in Mokhtar et al., 2017) is the first to use von Mises distribution $\mathrm{VM}(\mu,\kappa)$ to study the deviations of atomic weights from integer values. Specifically in this model, we focused on parameter estimation of von Mises distribution. The probability density function of von Mises distributions is given by

$$f(\theta;\mu;\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, \tag{1}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, can be described by:

$$I_o(\kappa)=\frac{1}{2\pi}\int_0^{2\pi}e^{\kappa\cos(\theta-\mu)}d\theta,\ \lambda=\frac{\nu}{\kappa} \tag{2}$$

where μ is the mean direction and κ is the concentration parameter.

The κ will influence the distribution and a concentrated von Mises distribution will have a high concentration parameter. While low concentration parameter for a dispersed von Mises distribution (Mokhtar et al. 2017). Meanwhile, the distribution approach to the normal distribution with the mean and variance as concentration parameter κ increase (Fisher, 1993) and (Mardia, 1972).

2. Materials and Methods

2.1 Circular Simultaneous Functional Relationship Model for Unequal Error Variances

Over the years, circular data have been developed based on circular theories and statistical methods (Mohamed et al. 2016). There are limited references by functional relationships for circular variables due to the complexity of the circular data as stated (Hassan et al., 2010, as cited in Badarisam et al., 2022). In 2014, the model of Circular Functional Relationship Model have been proposed by Satari et al. where it is an extension model from the Down Mardia (DM) Circular Regression Model to the error in variables model. Satari's model involves the study of the relationship between two circular variables. Next, Anuar (2018) extended the model from Satari et al. to become Circular Simultaneous Functional Relationship Model for Equal Variances and involves more than two circular variables.

Here, the proposed model of the Circular Simultaneous Functional Relationship Model for Unequal Variances which is an extension of Anuar's model. Specifically, Anuar's model of the ratio of error concentration parameter is equal, where $\lambda=1$. Theoretically, this model is relatively the same as Anuar's, but in this study, the ratio of error concentration parameters is unequal $\lambda=\frac{\nu}{\kappa}$ where λ can be extended to case when $\lambda_q=\lambda$. Therefore, in this model, without loss of generality, the value of $\lambda_q=$ 0.8 and 1.2 is fixed and known. Basically in this model, we assume that the observations (x_i, y_{ij}) are measured with errors $(\delta_i,\varepsilon_{ij})$. The errors are assumed to be independently distributed with von Mises distribution with $\delta_i\sim VM(0,\kappa)$ and $\varepsilon_i\sim VM(0,\nu)$ respectively. Suppose the Circular Simultaneous Functional Relationship Model for Unequal Variances is written as

$$x_i=X_i+\delta_i \text{ and } y_{ij}=Y_i+\varepsilon_{ij} \tag{3}$$

$$Y_{ij} = \beta_i + 2\tan^{-1}\left\{\omega_j \tan\left(\frac{X_i - \alpha_j}{2}\right)\right\} \tag{4}$$

where Y_{ij} is dependent random angle, ω_j is a slope parameter in the close interval $[-1,1]$, α_j and β_j are angular or location parameters and X_i is the independent random angle. In this model, two equations model simultaneously considered and there are $(n+3q+1)$ parameters estimated and those parameters $\alpha_1,..,\alpha_q,\beta_1,...,\beta_q,\omega_1,...,\omega_q,\kappa$ and incidental parameters $X_1, X_2,..., X_n$. Thus, the parameter estimation of the model are $\alpha_1,\alpha_2,\beta_1,\beta_2,\omega_1,\omega_2$ and κ.

2.2 Estimation of Parameters

The method applied for angular α and β and slope ω parameters are the *ms* function. The second method of the *polyroot* function is applied to the concentration parameter, κ and both methods are provided in SPLUS statistical package. Previously, Satari et al. (2014) applied the same method of parameter estimation in their model. Therefore, they found that the estimation for the circular model functional relationship model cannot be obtained in mathematically closed form using the maximum likelihood estimator method. Then, a similar case to our proposed model of circular simultaneous functional relationship model for unequal variances. This is due to the solution of the log-likelihood function is quite cumbersome. Hence, the estimation of parameters is obtained by minimizing the negative value of the log-likelihood function of the model numerically. The log-likelihood function for a given observations $(x, y_1, y_2) = ((x, y_{11}, y_{21})),....,((x_n, y_{1n}, y_{2n}))$ may be given in equation (5).

$$\begin{aligned}&\text{Log}\, L(\alpha_1,...,\alpha_q,...,\beta_1,...,\beta_q,...,\omega_1,...,\omega_q,\kappa, X_1,...,X_n \mid \lambda,(x, y_1, y_2)) = \\ &-2n\log[2\pi] - n(1+q)\log[I_o(\lambda\kappa)] + \sum_{i=1}^{n}\kappa\cos(x_i - X_i) + \\ &\sum_{j=1}^{q}\sum_{i=1}^{n}\lambda\kappa\cos\left(y_{ij} - \beta_j + 2\tan^{-1}\left\{\omega_j \tan\left(\frac{X_i - \alpha_j}{2}\right)\right\}\right).\end{aligned} \tag{5}$$

The first partial derivatives from the equation (5):

$$\begin{aligned}\frac{\partial \log L}{\partial\kappa} &= -n(1+q)\frac{I_o'(\kappa)}{I_o(\kappa)} - n(1+q)\lambda\frac{I_o'(\lambda\kappa)}{I_o(\lambda\kappa)} + \\ &\left(\sum_{i=1}^{n}\cos(x_i - X_i) + \lambda\sum_{i=1}^{q}\sum_{j=1}^{n}\cos\left(y_{ij} - \beta_j + 2\tan^{-1}\left\{\omega_j \tan\left(\frac{X_i - \alpha_j}{2}\right)\right\}\right)\right)\end{aligned} \tag{6}$$

$$\begin{aligned}&= -n(1+q)A(\kappa) - n(1+q)\lambda A(\lambda\kappa) + \\ &\left(\sum_{i=1}^{n}\cos(x_i - X_i) + \lambda\sum_{i=1}^{q}\sum_{j=1}^{n}\cos\left(y_{ij} - \beta_j + 2\tan^{-1}\left\{\omega_j \tan\left(\frac{X_i - \alpha_j}{2}\right)\right\}\right)\right)\end{aligned} \tag{7}$$

and by setting equation (7) = 0, we obtain

$$A(\kappa)+\lambda A(\lambda\kappa)=\frac{1}{n(1+q)}\left[\left(\sum_{i=1}^{n}\cos(x_i-X_i)+\lambda\sum_{i=1}^{q}\sum_{j=1}^{n}\cos\left(\begin{matrix}y_{ij}-\beta_j+2\tan^{-1}\\ \left\{\omega_j\tan\left(\frac{X_i-\alpha_j}{2}\right)\right\}\end{matrix}\right)\right)\right] \tag{8}$$

Then, the maximum likelihood estimator $\hat{\rho}$ of the precision parameter ρ is defined specifically

$$\hat{\rho}\left(\hat{\alpha}_1,...,\hat{\alpha}_q,\hat{\beta}_1,...,\hat{\beta}_q,\hat{\omega}_1,...,\hat{\omega}_q,\hat{\kappa},\hat{X}_1,...,\hat{X}_q\right)$$
$$=\frac{1}{n(1+q)}\left[\left(\sum_{i=1}^{n}\cos(x_i-X_i)+\lambda\sum_{i=1}^{q}\sum_{j=1}^{n}\cos\left(\begin{matrix}y_{ij}-\beta_j+2\tan^{-1}\\ \left\{\omega_j\tan\left(\frac{X_i-\alpha_j}{2}\right)\right\}\end{matrix}\right)\right)\right] \tag{9}$$

Nevertheless, the estimated parameters for α,β and ω used numerical approximation, *ms* function with a given set of initial values. The initial values are imposed in order to correspond to the maximum value of a precision parameter, ρ to the *ms* function. These initial values are obtained by calculating ρ for all possible pairs α,β,ω and $\hat{X}_1,...,\hat{X}_n$ in pre-specified sets of parameters values $\alpha=[-\pi,\pi],\beta=[-\pi,\pi],\omega=[-1,1]$ and $X_{10}=x_1,....,X_{n0}=x_n$. The concentration parameter, κ is approximated using the proposed roots of a polynomial function by Satari et al. (2014) with the proposed threshold of kappa, $A_s(\kappa)$ is for small κ and $A_L(\kappa)$ for large κ. For $A_s(\kappa)=\bar{R}$, we have

$$\frac{\kappa}{2}-\frac{\kappa^3}{16}+\frac{\kappa^5}{96}=\bar{R} \tag{10}$$

For $A_L(\kappa)=\bar{R}$, we get

$$1-\frac{1}{2\kappa}-\frac{1}{8\kappa^2}+\frac{1}{8\kappa^3}=\bar{R} \tag{11}$$

Both equations (10) and (11) are solved using the *polyroot* function in the Splus statistical package to find the roots of a complex polynomial equation.

2.3 Simulation Studies

The simulation study has been carried out and repeated 5000 times using SPlus statistical package to assess the efficiency of the proposed model, the Circular Simultaneous Functional Relationship Model for Unequal Variances. Several random samples of size n = 20, 60, 100 and 140 are generated respectively for circular variable X along with circular errors δ and ε using von Mises distribution. The generated random sample for circular variables X follows the values of $VM\left(\frac{\pi}{2},2\right)$ and is assumed fixed. Next, for circular errors, the values are obtained from $VM(0,\kappa)$ and $VM(0,\nu)$ where $\kappa\lambda=\nu$ with a set of concentration parameter values of $\kappa=1,5,15,20$ and a set of ratio concentration parameter, without loss of generality, λ= 0.8 and 1.2 respectively. Then, the observed values of variables x and y are generated using the generated random sample values, with fixed values of $\alpha_1=0.785$, $\alpha_2=0.524$, $\beta_1=0.785$, $\beta_2=0.524$ and for both ω is fixed values of $\omega=0.5$. The biasness measures of $\alpha_1,\alpha_2,\beta_1,\beta_2$ are circular mean, circular distance and mean resultant length. While the biasness measures of ω_1, ω_2 and κ are mean, absolute estimated bias (AEB) and estimated root mean square errors (ERMSE). Following are the measures:

a) Let θ be a generic for both α and β:

i. Circular mean,

$$\bar{\hat{\theta}} = \begin{cases} \tan^{-1}\left(S/C\right), & \text{if } S > 0, C > 0, \\ \tan^{-1}\left(S/C\right) + \pi, & \text{if } C > 0, \\ \tan^{-1}\left(S/C\right) + 2\pi, & \text{if } S < 0, C < 0, \end{cases}$$

where $C = \sum_{i=1}^{n} \cos\theta_i$ and $S = \sum_{i=1}^{n} \sin\theta_i$.

ii. Circular distance, $d = \pi - |\pi - |\bar{\hat{\theta}} - \theta||$

iii. Mean resultant length, $\bar{R} = 1/s\sqrt{\left(\sum\cos\left(\hat{\theta}_j\right)\right)^2 + \left(\sum\sin\left(\hat{\theta}_j\right)\right)^2}$

b) Let m be a generic for both ω and κ:

i. Mean, $\bar{\hat{m}} = 1/s\sum_{j=1}^{s} \hat{m}_j$

ii. Absolute estimated bias, $AEB = |\bar{\hat{m}} - m|$

iii. Estimated root mean square errors, $ERMSE = \sqrt{1/s\sum_{j=1}^{s}\left(\hat{m}_j - m\right)^2}$

3. Results and Discussion

Tables 1, 2, 3, and 4 represent the simulation results for the angular parameters, $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$ with the ratio of concentration parameter, $\lambda = 0.8$. The trend for the circular mean and the circular distance gets smaller as the *n* value increases for a fixed *κ*. The mean value of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$ are close to the true value of $\alpha_1 = 0.785$, $\alpha_2 = 0.524$, $\beta_1 = 0.785$, $\beta_2 = 0.524$ as the *n* value increases. Besides, Tables 5 and 6 represent the slope, $\hat{\omega}$ and concentration parameters, κ with $\lambda = 0.8$. We can see that the estimation means for both tables are close to the true value of $\hat{\omega} = 0.5$. The AEB and ERMSE both show a clear decreasing pattern as the *n* value increases for a fixed κ . The same results can be shown from Tables 7,8,9,10,11 and 12 for the ratio of concentration parameter, $\lambda = 1.2$. Besides, as the ratio of concentration parameter increases, the results for all parameters show the consistency in the value of biasness. In summary, we note that as *n* values increase for any fixed κ *,* the parameter estimation is closer to the true value and the bias measures decrease.

Table 1. Simulation results of $\hat{\alpha}_1$ when $\lambda = 0.8$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.9932	0.6078	0.8687
	60	0.8807	0.6053	0.9086
	100	0.8337	0.5483	0.9548
	140	0.8127	0.4373	0.9813
5	20	0.9823	0.3269	0.8610
	60	0.8827	0.3033	0.8709
	100	0.8119	0.2895	0.9483
	140	0.8006	0.1972	0.9887
15	20	0.9416	0.1562	0.8539
	60	0.9155	0.1301	0.9247
	100	0.8068	0.1214	0.9569
	140	0.7891	0.0967	0.9889
20	20	0.9010	0.1156	0.8934
	60	0.7957	0.1103	0.9812
	100	0.7808	0.0954	0.9849
	140	0.7852	0.0500	0.9922

Table 2. Simulation results of $\hat{\alpha}_2$ when $\lambda = 0.8$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.8390	0.5154	0.8025
	60	0.7787	0.3732	0.8897
	100	0.6959	0.2551	0.9004
	140	0.5477	0.0091	0.9511
5	20	0.6520	0.8084	0.8163
	60	0.5708	0.6172	0.8971
	100	0.5542	0.3016	0.9493
	140	0.5299	0.1063	0.9741
15	20	0.5554	0.6218	0.8248
	60	0.5421	0.5885	0.9032
	100	0.5306	0.3070	0.9520
	140	0.5253	0.0217	0.9874
20	20	0.5461	0.6525	0.8651
	60	0.5318	0.4282	0.9260
	100	0.5261	0.2825	0.9623
	140	0.5245	0.0246	0.9914

Table 3. Simulation results of $\hat{\beta}_1$ when $\lambda = 0.8$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.9323	0.4380	0.9008
	60	0.8725	0.4271	0.9468
	100	0.8303	0.3449	0.9604
	140	0.7955	0.3300	0.9862
5	20	0.8877	0.2317	0.9330
	60	0.8643	0.2189	0.9683
	100	0.8213	0.2059	0.9861
	140	0.7869	0.1018	0.9920
15	20	0.8119	0.1065	0.9624
	60	0.7971	0.0997	0.9820
	100	0.7952	0.0912	0.9910
	140	0.7854	0.0825	0.9945
20	20	0.8009	0.0806	0.9725
	60	0.7881	0.0787	0.9864
	100	0.7871	0.0717	0.9918
	140	0.7852	0.0498	0.9943

Table 4. Simulation results of $\hat{\beta}_2$ when $\lambda = 0.8$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.6858	0.4094	0.8985
	60	0.6068	0.3388	0.8997
	100	0.5848	0.3168	0.9770
	140	0.5642	0.3062	0.9872
5	20	0.6582	0.404	0.9294
	60	0.5986	0.3749	0.9697
	100	0.5487	0.3250	0.9799
	140	0.5296	0.2654	0.9890
15	20	0.5915	0.3816	0.9613
	60	0.5716	0.3542	0.9795
	100	0.5394	0.302	0.9898
	140	0.5270	0.2321	0.9905
20	20	0.5900	0.3757	0.9791
	60	0.5621	0.349	0.9896
	100	0.5366	0.2986	0.9928
	140	0.5246	0.2282	0.9933

Table 5. Simulation results of $\hat{\omega}$ when $\lambda = 0.8$

Concentration parameter, κ	n	Mean	Absolute Estimated Bias (AEB)	Estimated Root Mean Square Error (ERMSE)
1	20	0.4470	0.4886	0.4977
	60	0.4603	0.4623	0.4798
	100	0.4670	0.4553	0.3913
	140	0.4684	0.3300	0.3700
5	20	0.4722	0.1262	0.2137
	60	0.4756	0.1144	0.1649
	100	0.4780	0.1078	0.1325
	140	0.4808	0.1042	0.1249
15	20	0.4855	0.0559	0.1218
	60	0.4865	0.0522	0.0940
	100	0.4878	0.0505	0.0763
	140	0.4881	0.0485	0.0670
20	20	0.4897	0.0393	0.0511
	60	0.4963	0.0300	0.0407
	100	0.4971	0.0273	0.0330
	140	0.4992	0.0221	0.0208

Table 6. Simulation results of $\hat{\kappa}$ when $\lambda = 0.8$

Concentration parameter, κ	n	Mean	Absolute Estimated Bias (AEB)	Estimated Root Mean Square Error (ERMSE)
1	20	1.5250	0.3700	1.4755
	60	1.3609	0.2391	1.4692
	100	1.2385	0.2221	0.0850
	140	1.1722	0.1725	0.4689
5	20	5.6839	0.3750	3.2402
	60	5.4310	0.3369	2.2311
	100	5.2810	0.3190	1.2267
	140	5.0039	0.3161	0.2258
15	20	14.9489	0.5511	8.1962
	60	14.6680	0.5320	6.1916
	100	14.2049	0.4251	5.1867
	140	14.0499	0.3101	4.1809
20	20	21.0114	0.8886	10.1886
	60	20.8195	0.7805	7.6621
	100	20.5731	0.5269	5.1805
	140	20.1342	0.3658	3.1734

Table 7. Simulation results of $\hat{\alpha}_1$ when $\lambda = 1.2$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.9811	0.6064	0.8721
	60	0.8707	0.6003	0.9165
	100	0.8235	0.5312	0.9632
	140	0.8127	0.4178	0.9800
5	20	0.9703	0.3069	0.8730
	60	0.8218	0.2899	0.8894
	100	0.8100	0.2716	0.9713
	140	0.7996	0.1942	0.9877
15	20	0.9361	0.1422	0.8739
	60	0.9025	0.1206	0.9377
	100	0.8000	0.1195	0.9610
	140	0.7863	0.0847	0.9929
20	20	0.9110	0.1065	0.9134
	60	0.8897	0.1000	0.9722
	100	0.8418	0.0852	0.9850
	140	0.7849	0.0400	0.9982

Table 8. Simulation results of $\hat{\alpha}_2$ when $\lambda = 1.2$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.8390	0.5154	0.8025
	60	0.7787	0.3732	0.8897
	100	0.6959	0.2551	0.9004
	140	0.5477	0.0091	0.9511
5	20	0.6520	0.8084	0.8163
	60	0.5708	0.6172	0.8971
	100	0.5542	0.3016	0.9493
	140	0.5299	0.1063	0.9741
15	20	0.5554	0.6218	0.8248
	60	0.5421	0.5885	0.9032
	100	0.5306	0.3070	0.9520
	140	0.5253	0.0217	0.9874
20	20	0.5461	0.6525	0.8651
	60	0.5318	0.4282	0.9260
	100	0.5261	0.2825	0.9623
	140	0.5245	0.0246	0.9914

Table 9. Simulation results of $\hat{\beta}_1$ when $\lambda = 1.2$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.9431	0.4370	0.8998
	60	0.8852	0.4251	0.9384
	100	0.8430	0.3339	0.9516
	140	0.7958	0.3210	0.9726
5	20	0.8764	0.2215	0.9230
	60	0.8513	0.2090	0.9538
	100	0.8113	0.1599	0.9711
	140	0.7864	0.1024	0.9800
15	20	0.8125	0.1055	0.9542
	60	0.7961	0.0987	0.9810
	100	0.7940	0.0903	0.9901
	140	0.7853	0.0812	0.9954
20	20	0.7999	0.0776	0.9748
	60	0.7884	0.0631	0.9884
	100	0.7862	0.0546	0.9980
	140	0.7850	0.0338	0.9997

Table 10. Simulation results of $\hat{\beta}_2$ when $\lambda = 1.2$

Concentration parameter, κ	n	Circular Mean	Circular Distance	Mean Resultant Length
1	20	0.6782	0.4072	0.8990
	60	0.6211	0.3265	0.9602
	100	0.5734	0.3184	0.9755
	140	0.5602	0.3026	0.9889
5	20	0.6428	0.4033	0.9314
	60	0.5880	0.3708	0.9682
	100	0.5451	0.3150	0.9795
	140	0.5260	0.2554	0.9893
15	20	0.5812	0.3761	0.9701
	60	0.5744	0.3527	0.9856
	100	0.5304	0.2999	0.9879
	140	0.5269	0.2215	0.9945
20	20	0.5700	0.3621	0.9761
	60	0.5531	0.3384	0.9836
	100	0.5259	0.2758	0.9910
	140	0.5241	0.2109	0.9964

Table 11. Simulation results of $\hat{\omega}$ when $\lambda = 1.2$

Concentration parameter, κ	n	Mean	Absolute Estimated Bias (AEB)	Estimated Root Mean Square Error (ERMSE)
1	20	0.4560	0.4875	0.4861
	60	0.4731	0.4531	0.4413
	100	0.4770	0.4486	0.3579
	140	0.4885	0.2989	0.3300
5	20	0.4735	0.1226	0.2127
	60	0.4793	0.1173	0.1638
	100	0.4868	0.1054	0.1314
	140	0.4880	0.1002	0.1259
15	20	0.4862	0.0459	0.1118
	60	0.4870	0.0432	0.0830
	100	0.4889	0.0406	0.0663
	140	0.4900	0.0386	0.0579
20	20	0.4879	0.0281	0.0406
	60	0.4885	0.0100	0.0371
	100	0.4907	0.0164	0.0340
	140	0.4983	0.0137	0.0218

Table 12 Simulation results of $\hat{\kappa}$ when $\lambda = 1.2$

Concentration parameter, κ	n	Mean	Absolute Estimated Bias (AEB)	Estimated Root Mean Square Error (ERMSE)
1	20	1.6450	0.3605	1.4765
	60	1.4719	0.2318	1.4683
	100	1.3385	0.2214	0.4595
	140	1.1102	0.1025	0.1078
5	20	5.6739	0.3500	3.2130
	60	5.3300	0.3196	2.2011
	100	5.2744	0.2990	1.1674
	140	5.0062	0.1147	0.1285
15	20	14.8189	0.5538	8.1762
	60	14.5920	0.5360	6.1804
	100	14.3046	0.4241	5.1377
	140	14.0477	0.3001	4.1002
20	20	21.0120	0.8784	10.1876
	60	20.7954	0.7705	7.6281
	100	20.3618	0.5496	5.1405
	140	20.1146	0.3648	3.1934

3.1 Application of The Model Using Real Circular Data

The model is illustrated by using two different sets of wind and wave direction data with two different of ratio concentration parameter, $\lambda = 0.8$ and 1.2. The first data set consists of 49 observations and collected from the Humberside coast of the North Sea, United Kingdom. The wind direction data, as variable x and the wave direction are addressed as the variables y_1 and y_2 measured by anchored wave buoy and HF radar. The second data set is collected from Bayan Lepas airport and consists of 62 observations. The data were recorded at two different pressures. The variable x recorded at pressure 850 Hpa with 5000 m height. While for variable y_1 at pressure 1000 Hpa with 300 m height and y_2 at pressure 500 Hpa with 1900m height.

Tables 13, 14, 15 and 16 show the parameter estimates and the standard error of parameter estimates for both real data sets. We can say that the standard error for the parameter estimates for both data sets is small, and the concentration parameter value is high. The higher the value of the concentration parameter, the closer the measurement data to each other. Further, both ratio of concentration parameter, $\lambda = 0.8$ and 1.2 for two different data sets show the results consistently where the standard error gets the small value, respectively.

Table 13. Parameter estimation of Humberside wind and wave direction data when $\lambda = 0.8$

DATA	WAVE	
n	49	
Parameter	Estimate	Standard Error
$\hat{\alpha}_1$	2.2296	0.2774
$\hat{\alpha}_2$	2.6520	0.2852
$\hat{\beta}_1$	1.9973	0.1997
$\hat{\beta}_2$	2.7717	0.2230
$\hat{\omega}_1$	0.1004	0.0680
$\hat{\omega}_2$	0.9801	0.7628
$\hat{\kappa}$	10.5662	0.1471

Table 14. Parameter estimation of Bayan Lepas wind and wave direction data when $\lambda = 0.8$

DATA	BAYAN LEPAS	
n	62	
Parameter	Estimate	Standard Error
$\hat{\alpha}_1$	2.2600	0.2573
$\hat{\alpha}_2$	2.2760	2.2503
$\hat{\beta}_1$	1.0584	0.2007
$\hat{\beta}_2$	1.7858	0.1943
$\hat{\omega}_1$	0.0846	0.0399
$\hat{\omega}_2$	0.0154	0.0336
$\hat{\kappa}$	11.8115	0.7767

Table 15. Parameter estimation of Humberside wind and wave direction data when $\lambda = 1.2$

DATA	WAVE	
n	49	
Parameter	Estimate	Standard Error
$\hat{\alpha}_1$	2.7918	0.3352
$\hat{\alpha}_2$	2.4011	0.4370
$\hat{\beta}_1$	2.9389	0.2548
$\hat{\beta}_2$	2.0021	0.2080
$\hat{\omega}_1$	0.3581	0.8092
$\hat{\omega}_2$	0.0105	0.0408
$\hat{\kappa}$	9.7830	0.1471

Table 16. Parameter estimation of Bayan Lepas wind and wave direction data when $\lambda = 1.2$

DATA	BAYAN LEPAS	
n	62	
Parameter	Estimate	Standard Error
$\hat{\alpha}_1$	2.2856	0.4981
$\hat{\alpha}_2$	0.2452	0.5494
$\hat{\beta}_1$	0.8946	0.3625
$\hat{\beta}_2$	0.2718	0.3844
$\hat{\omega}_1$	0.0465	0.2042
$\hat{\omega}_2$	0.0576	0.9203
$\hat{\kappa}$	10.0287	0.4544

4. Conclusion

The new proposed model of Circular Simultaneous Functional Relationship Model for Unequal Variances is the extended model of Circular Simultaneous Functional Relationship Model for Equal Variances as proposed by Anuar (2018). The illustration of real wind and wave data sets is given to show its practical applicability and the simulation results for all parameter estimates show a good result. This may be indicated that the proposed method works well with the proposed model as it provides good estimates.

5. Acknowledgements

This research is financially supported by the National Defence University of Malaysia, Fundamental Research Grant Scheme (FRGS), FRGS/1/2019/STG06/UPNM//1.

6. References

Ahmad, N., Nawawi, M.S., Zainuddin, M.Z, Nasir, Z.M., Yunus, R. M., & Mohamed., I. 2020. A New Crescent Moon Visibility Criteria using Circular Regression Model: A Case Study of Teluk Kemang, Malaysia. *Sains Malaysiana* 49(4): 859-870

Anuar, M. M. A. (2018). On Parameter Estimation, Confidence Intervals and Outlier Detection for Some Circular Models, PhD. Thesis, Universiti Pertahanan Nasional Malaysia.

Badarisam, F., & Rambli, A., & Sidik, M. (2020). A Comparison on Two Discordancy Tests to Detect Outlier in Von Mises (VM) Sample. *Indonesian Journal of Electrical Engineering and Computer Science*. 19. 156.

Badarisam, F., & Anuar M., M., S., & Hussin, A., & Rambli, A., & Zulkifli, Nurul. (2022). Confidence Interval for Parameters Estimates in Circular Simultaneous Functional Relationship Model (CSFRM) for Equal Variances using Normal Asymptotic and Bootstrap Confidence Intervals. *Sains Malaysiana*. 51. 3819-3827.

Caires, S. and Wyatt, L. R. (2003). A Linear Functional Relationship Model for Circular Data with an Application to the Assessment of Ocean Wave Measurement. American Statistical Association and the Internal Biometric Society *Journal of Agricultural, Biological and Environmental Statistics*: 8 (2),153-169.

Fisher N. 1993. *Statistical Analysis of Circular Data.* Cambridge University Press, Cambridge.

Gatto, R. and Jammalamadaka, S. R. (2007). The Generalized Von Mises Distribution. *Statistical Methodology* 4, 341-353.

Hassan, S. F. (2010). Development of Simultaneous Linear Circular Functional Relationship Model and Analysis of Missing Values, PhD Thesis, Universiti Pertahanan Nasional Malaysia.

Hassan, S.F., Hussin, A.G. & Zubairi, Y.Z. 2012. Improved Efficient Approximation of Concentration Parameter and Confidence Interval For Circular Distribution. *ScienceAsia* 38: 118-124.

Hussin, A. G., Fieller, N. R. J. and Stillman, E. C. 2004. Linear Regression for Circular Variables with Application to Directional Data, *Journal of Applied Science & Technology (JAST)*: 9 (1 and 2), 1-6.

Jammalamadaka, S. R., Bhadra, N., Chaturvedi, D., Kutty, T. K., Majumdar, P. P. and Poduval, G. (1986). *Functional Assessment of Knee and Ankle During Level Walking*. In Matusita, K., editor. Data Analysis in Life Science, 21-54. Indian Statistical Institute, Calcutta, India.

Jammaladaka, S.R. & Sengupta, A. 2001. *Topics in Circular Statistics*. World Scientific Publishing.

Johnson, R. A. & Wehrly, T. (1978). Some Angular-Linear Distributions and Related Regression Models. *J. Am. Statist. Assoc*. 73, 602-6.

Jones, T. A. and James, W. R. 1969. Analysis of Bimodal Orientation Data. *Mathematical Geology* 1 (2), 129-135.

Mardia, K. V. 1972. *Statistics of Directional Data*. London: Academic Press.

Mardia, K. V. and Jupp, P. E. (2000). *Directional Statistics*, John Wiley & Sons.

Mohamed, I. B., Rambli, A., Khaliddin, N., & Ibrahim, A. I. N. 2016. A New Discordancy Test in Circular Data using Spacings Theory, *Communications in Statistics - Simulation and Computation* 45:8, 2904-2916

Mokhtar, N. A. & Zubairi, Y. & Hussin, A. (2017). Estimation of Concentration Parameter for Simultaneous Circular Functional Relationship Model Assuming Unequal Error Variance. *Sains Malaysiana*. 46. 1347-1353. 10.17576/jsm-2017-4608-22.

Rivest, L. P. (1997). *A Decentred Predictor for Circular –Circular Regression*. Biometrika, 84(3), 717-726.

Satari, S. Z., Hussin, A. G., Zubairi, Y. Z. and Hassan, S. F. (2014). A New Functional Relationship Model for Circular Variables, *Pakistan Journal of Statistics*: 30(3), 397-410.

von Mises, R. 1918. Uber die “Ganzzahligkeit” der Atomgewicht und verwandte Fragen. *Physikalischez.* 19: 490-500.