MALAYSIAN PRE-SERVICE MATHEMATICS TEACHER'S COVARIATIONAL REASONING: A VASE PROBLEM

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Abstract: Recently, abundant of studies were conducted on covariational reasoning, indicating its importance in understanding the idea rate of change. Despite that, the concept of covariational reasoning was also urged by past researchers to call out the importance for students to conceive a relationship between quantities. This is because past studies disclosed the students, either in secondary or tertiary level, a lack of understanding in conceiving how quantities influenced each other. Thus, the researchers took this initiative to investigate on what kinds of covariational reasoning was done by Malaysian pre-service mathematics teachers, due to its importance in conceiving dynamical activity in real-life application. This study was participated by four pre-service mathematics teachers which was selected through purposive sampling to ensure that rich data was obtained. The findings revealed that the participants' covariational reasoning was underdeveloped in many ways. The result was consistent with other studies as it was found that pre-service mathematics teachers were unable to conceive variables based on given situation, since they were either interchanged roles between variables or setting time as the input. In return, most of the participants were unable to conceive a smooth covariation between variables. The result displayed that only one out of four participants in this study are able to coordinate the variables successively. Hence, this implies that the difficulties in covariational problem were still unattended, especially in conceiving how a relationship works among the changed quantities under a situation.

Keywords: Covariational Reasoning, Rate of Change, Pre-Service Mathematics Teachers

INTRODUCTION

Rate of change is one of the most fundamental concepts in calculus (Thompson, 1994) and is used to describe the relationship between the changing quantities (Tyne, 2016). Since the world is dynamic in nature, thus everything is always changing. All these changes can be explained using the notion of rate of change. Some of them may alter their behaviour dynamically and some may display cyclic pattern (Marsitin, 2019). Calculus is used to help us understand this changing behaviour and their complex characteristics. Using this advantage, a lot of things could be predicted to produce an informative data which is beneficial for the sake of human life.

However, students were found to be lacking in interpreting rate of change as dynamic process (Habre & Abboud, 2006; Monk, 1994; Thompson, 1994). This is due to the weak understanding of function as well as the difficulty in modelling the functional relationship (Monk, 1992; Thompson, 1994). Lack of understanding in functional relationship may prevent students from perceiving the dynamic view of function which will then lead to difficulties in understanding the idea of rate of change. Thus, the researchers in their studies (e.g., Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Thompson, 1992) have developed the idea of covariation which became an essential element to perceive rate of change as a functional relationship. Covariation reasoning is an approach to apprehend the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 357). However, past studies have revealed that university students and pre-service mathematics teachers have trouble to perform well in covariation tasks (Carlson et al., 2002; Kertil, 2014). This reflected their weak knowledge in rate of change since they were unable to described and apprehend it as a functional relationship.

Moreover, the difficulties faced by both teachers or students either at secondary or tertiary level were mainly highlighted in coordinating covarying quantities (e.g. Carlson et al., 2002; Kertil et al., 2019; Lynn et al., 2017; Mkhatshwa, 2020; Paoletti & Moore, 2017; Yemen et al., 2017). This is because covariational reasoning was about dealing with two quantities that changes tandemly, hence, creating a relationship between these two quantities.

The students' difficulties in covariational reasoning were believed to underlie this quantifying problem, since their cognitive mind is unable to visualize how two quantities relate with each other (Johnson, 2015; Mkhatshwa, 2020; Thompson & Carlson, 2017). Hence, this indicates that this kind of problem was closely associated with another idea of quantitative reasoning (Mkhatshwa, 2020; Thompson, 2011). The reason was that quantitative reasoning is an analysed skill under problem context in terms of quantities and relationship among involved quantities (Thompson, 1993). Thompson (2011) had mentioned the process of quantifying included a relationship between the quantities, hence, shifted his attention to discover how students conceive covariational reasoning situations as "composed of quantities and relationships among quantities whose values vary" (Thompson & Carlson, 2017, pp. 424–425). Prior to that, Castillo-garsow (2012) addressed about inducing continuous quantitative reasoning as another important skill in covariational reasoning. Indeed, Johnson (2015) stated that students are able to come to understand about instantaneous rate of change by understanding average rate of change. The result of her study showed that participated students were able to reason quantitatively as they are able to envision the completed amounts of change in height and volume for different bottles sections. Hence, it may become a root for the students in reasoning on instantaneous rate of change, thus, envisioning changes as a continuing process (Johnson, 2015). Due to that, the essence in covariational reasoning as emphasized by other researchers was also to make sense of the relationship among quantities under context (Castillo-garsow, 2012; Johnson, 2015; Mkhatshwa, 2020), besides capability of conceiving the individual attribute varied and those attributes concurrently varied (Carlson et al., 2002; Thompson & Carlson, 2017).

Eventually, understanding in covariation process is important to help students to perceive how a relationship works between two quantities simultaneously, hence, able to conceive the relationship among changing quantities. The discussion and findings of these past studies (e.g. Carlson et al., 2002; Kertil et al., 2019; Lynn et al., 2017; Mkhatshwa, 2020; Paoletti & Moore, 2017; Yemen et al., 2017) have indicated that there exists this problematic aspect in terms of knowledge of rate of change among students and pre-service teachers. As a result, this issue causes the researcher to wonder upon covariational reasoning had by Malaysian pre-service mathematics teachers, since this will help to contribute to the body of knowledge of calculus learning in Malaysia and foster a better understanding about the idea of rate of change that these pre-service teachers had, so that these teachers could deliver a correct idea of calculus in transferring the knowledge to their students later. Thus, this is why the present study intends to explore about covariational reasoning among pre-service mathematics teachers, since acknowledging the covariation process between two variables is another approach in dealing about how rate of change works (Carlson et al., 2002; Kertil, 2014; Saldanha & Thompson, 1998). Therefore, a qualitative research methodology will be employed to enable an investigation for covariational reasoning by Malaysian pre-service mathematics teachers.

THEORETICAL FRAMEWORK

Covariational reasoning plays with dynamic mental operation and it is one of the ways to describe pre-service teachers' understanding in rate of change since it can help to act as evidence in visualizing student teacher knowledge on the rate of change. It is considered as one of the best ways to understand student teachers' knowledge of rate of change by seeking into their explanation or reasoning skills (Carlson et al., 2002; Carlson et al., 2010; Thompson & Carlson, 2017). In this regard, the covariational reasoning framework by Carlson et al. (2002) was established and pioneered in order to facilitate the individual's knowledge in modelling functional relationship of a situation involving rate of change. Carlson et al. (2002) also defined covariational reasoning as a "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 357). This covariational reasoning ability gained by the individuals enable them to perceive the coordination of changes that simultaneously take place between the quantities. This is a crucial starting point to interpret rate of change as a functional relationship (Carlson et al., 2002; Çetin, 2009). Plus, this ability of reasoning is considered as a basis in calculus foundation since it acknowledges how rate of change works (Thompson, 1994).
In the framework provided by Carlson et al. (2002), five mental actions of covariational reasoning are shown in Table 1 below. The description of mental action and its behaviours are classified into the associated mental action that are demonstrated by student teacher as he or she engages in covariation tasks. Afterwards, the student teacher is sorted into the level of classification based on the overall image presented in order to support the mental actions that he or she exhibited in the covariation task. The level of covariation classification provided in Table 2 contains five distinct developmental levels. The student teacher's covariational reasoning ability is said to reach a given level of developmental when it supports the mental actions associated with that level and the actions associated with all its lower levels.

Table 1. Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviours
Mental action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labelling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of changes of the output while considering changes in the input
Mental action 4 (MA4)	Coordinating the average rate of change of the function with uniform increment of change in the input variable	• Constructing contiguous secant lines for the domain • Verbalizing the awareness of the rate of change of the output (while respect to the input) while considering uniform increments of the input
Mental action 5 (MA5)	Coordinating the instantaneous rate of change of the function in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

(Carlson et al., 2002, p. 357)

Table 2. Levels of the Covariation Framework

Covariational Reasoning Levels
Level 1 (L1). ***Coordination*** At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).
Level 2 (L2). ***Direction*** At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are both supported by L2 images.
Level 3 (L3). ***Quantitative Coordination*** At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images
Level 4 (L4). ***Average Rate*** At the average rate level, the images of coordination can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.
Level (L5). ***Instantaneous Rate*** At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes the awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

(Carlson et al., 2002, p. 358)

Subsequently, Thompson and Carlson (2017) had revised covariational reasoning framework by scrutinizing into how individual conceived varied quantities (Castillo-garsow, 2010, 2012), together with Thompson (2011) study of quantitative reasoning and study of coordination of quantities that changed (Carlson et al., 2002; Carlson et al.,

2003) as a recent construct to be used. There were six levels in Major Level Covariational Reasoning comprised by different of covariation and coordination as shown in Table 3 below. However, Thompson and Carlson (2017) also stated that concept of rate of change was not included in their recent framework as they only emphasized on how two quantities varied at different levels. The rationality behind that was they considered conceptualization of rate of change goes beyond covariational reasoning as it includes conceptualization of proportionality, ratio and accumulation (Thompson & Carlson, 2017). Despite that, Kertil et al. (2019) incorporated how students quantified rate of change in their study and managed to characterized students' covariational reasoning into three categories; i) identifying variables, ii) ways of coordinating the variables, and iii) quantifying the rate of change as shown in Table 4 below. They disclosed the study's outcome by adding new categories which were identifying variables and quantifying rate of change and emphasized these as significant features required in order to build students' skill in covariational reasoning.

Table 3. Major Levels of Covariational Reasoning

Level	**Description**
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

(Thompson & Carlson, 2017, p. 441)

Table 4. Categories, sub-categories, abbreviations and examples of coding schema used in analysing covariational reasoning

1. Identifying variables	
Thinking by primary variables	Values of quantities explicitly conceived in relation to each other as dependent and independent variables and labelled on a graph (e.g., In the Water Tank task, determining volume as independent variable and height as the dependent variable and labelling on a graph).
Thinking by secondary variables	Values of original quantities conceived in terms of a common parameter such as "time" (e.g., In the Water

	Tank task, taking "time" or "radius of cross sections" as the independent variable).
Reversing the roles	Values of quantities involved in a situation explicitly conceived in relation to each other, but with the roles of input and output variables reversed (e.g., In the Water Tank task, determining volume as the dependent variable and height as the independent variable).
2. Ways of coordinating the variables	
Uncoordinated way of thinking (corresponding to Thompson and Carlson's (2017) "no coordination" and "pre-coordination" levels)	Considering the covarying quantities as two separate functions with respect to a common parameter or considering the change in only one variable (e.g., "The radius of first reel increases rapidly and the radius of second reel decreases slowly by the time")
Indirect Coordination	Conceiving the functional dependency between covarying quantities depending on a third variable. Incapability of conceiving the invariant relationship between the covarying quantities by removing the common parameter (e.g., In the Water Tank task, using radius of cross-sections as an implicit independent variable; "Because the water tank is enlarging in an upwards direction, it can be said that the increase in height will gradually decrease when compared with the increase in volume").
Direct Coordination (corresponding to Thompson and Carlson's (2017) "gross coordination" and "coordination of values" levels)	Directly coordinating the changes in one quantity as happening simultaneously with changes in another quantity without focusing on the intensity of change as if there is a linear relationship (e.g., "The height increases as the volume increases").
Direct and Systematic Coordination (corresponding to Thompson and Carlson's (2017) "chunky" and "smooth continuous" covariation levels)	Envisioning changes in one quantity as happening simultaneously with changes in another quantity and being able to systematically change the input variable. Observing the simultaneous variation in the output variable, in either chunky continuous or smooth continuous forms (e.g., "For every equal amount of change in the distance, we have to observe the simultaneous change in height." (Chunky) and "Amount of change in height per unit volume" (Smooth)).
3. Quantifying the rate of change	
Gross quantification	Deciding perceptually without providing mathematical justification, or providing explanations involving inconsistent quantitative operations (e.g., "Because the trigonometric functions do not change linearly, I drew such a curved graph" and "As the radius increases, since the volume will increase slowly, the increase in height is slower").
Extensive quantification	Focusing on the successive changes in only one of the quantities. Changing the independent variable with constant and equal increments and additively comparing the simultaneous change in the dependent variable (e.g., "The height increases more for every equal amount of change in volume."). There is no ratio-based reasoning.
Intensive quantification	Uniformly changing the input variable (being aware that the unit can be selected as small as possible), and observing the simultaneous change in the output

	variable; or changing the input variable with constant and equal increments, and multiplicatively comparing the simultaneous change in the output variable (e.g., "The angle increases at a decreasing rate with respect to per unit height" and "The ratio of the increase in height to the increase in volume gradually gets bigger").

(Kertil et al., 2019, p. 5)

METHODOLOGY

This present study employed a qualitative research method since the interest of this research was to seek understanding of meaning by people, which were pre-service teachers' covariational reasoning of rate of change. This is also based on the literature and the problems gathered in rate of change which had led to posing the research question: what kinds of covariational reasoning had by Malaysian pre-service mathematics teachers in rate of change? This research question came forward to perceive pre-service mathematics teachers' covariational reasoning in rate of change. Knowledge, epistemologically, are believed to be gained differently by different people which served as a naturalistic aspect by the interpretivist. This matter is referred as a distinctive event and led to the use of inductive approach (Creswell, 2013). Hence, the best method to collect data is using qualitative methodology comprised by interview as a main technique, supplemented by observation and document analysis techniques. These techniques help to enable the pre-service mathematics teacher's covariational reasoning through verbal explanation and non-verbal behaviour, which are stored in the form of words, behaviour and written answer such as video, audio and documents. It will directly access into participant's covariation individually and help to descriptively understand their covariational skill in detail.

Therefore, to allow this type of investigation, a purposive sampling must be selected. This sampling will enable the sample to be collected from those who can learn the most and have a great deal about the central issue. To enable this investigation, the procedure of this study will follow qualitative nature study. Hence, interview is found to be the most appropriate technique to be used since this study aims to access pre-service teachers' covariational reasoning which can be classified as an unobservable entity conceived in individual's thought (Merriam, 1998). Hence, the type of interview that will be used in this study was task-based interview as this type has been widely used in mathematics education to gain knowledge about individual or group existing mathematical knowledge (Goldin, 2000; Maher & Sigley, 2014). In comparison with the conventional method of paper-and-pencil based test to answer mathematical tasks, the paper and pencil method will only limit the knowledge investigation and certainly inadequate in addressing the actual knowledge had by the individual (Goldin, 2000). This is why task-based interview is used because it will describe pre-service teacher knowledge by providing an insight into their mathematical solution, making it possible to focus directly to their knowledge comprehensively rather than just on the pattern of correct and incorrect answers they produced.

RESULTS

In the findings, there were four pre-service mathematics teachers consisting of Chen, Arya, Maryam and Johan and they exhibited different kinds of covariation in Vase Task (refer Appendix 1). In the beginning, Chen appeared to coordinate between water flows in constant with height of water, though his graph in Figure 1 below had written time as x-axis (pseudo-analytic MA1).

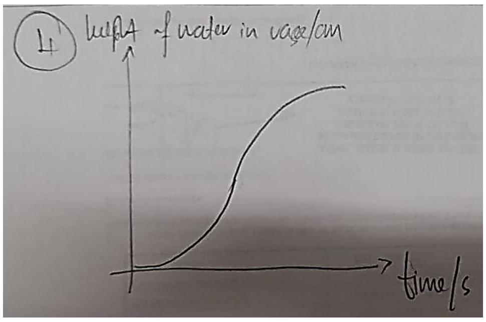

Figure 1

He firstly stated that if water moves in constantly, rate of height as well as speed of height was increased first and then decreased after passing through the middle part (MA5). This suggests that he perceived as water flows in constantly, the height of water increasing, (MA2) but changing with its rate.

“because I see the method ok, common sense that water, if time not changing, the water flows in constantly, so maybe speed of height will increase then become slower (MA5), it will fast then slow because from bottom, the surface area what… become less and up to above it increasing, so if follow like this, it supposedly I need to do urmm supposedly the graph will increase showing higher slope, ok, so it from flat goes to higher but after at the middle, because at middle up to above, it become bigger, so speed will become slower back, so it will reverse with another half (MA4).”

He includes the reason of changing in slope was due to the shape of vase at the middle, where in the first half, it narrows down causing the water that flows in constantly filled up the area rapidly, hence causing the height and rate to fill in increased rapidly (MA4). He was seen in here to coordinate between constant amount of water flow in with changes of height (MA3) and led to its changes of rate.

“Because the vase it become narrow, when it becomes narrow, if time and its speed of water flow into the vase constantly, so, supposedly the narrow area, it will cause the height, the rate like fill in the space rapidly (MA3).”

“it become increase then become decrease, let say the slope highest, for slope eh, for slope of the graph in first half, it was highest can reach 10, ok, let says, ok, so for same situation, when it goes to second half, the highest it can reach when it decreases starts from 10, it will not reach 11 then decreasing, I want convey is that its like symmetrical, like reflection.”

Hence, from his explanation he was appeared to coordinate correctly between constant amount of water flow in with changing of height with correct concavity and inflection point, as shown in his graph, though his graph was written time as x-axis. Plus, his explanation also did not mentioned how time influenced the changed of height but how constant flow of water was.

Nevertheless, in Vase Task, Arya also sketched a similar graph as Chen, but she added another concave-upward curve, which totalled up as three concavities in the graph as shown in Figure 2 below. She appeared to coordinate between height of water with amount of water that filling up at constant rate and it also shown in her label of axis in the graph below.

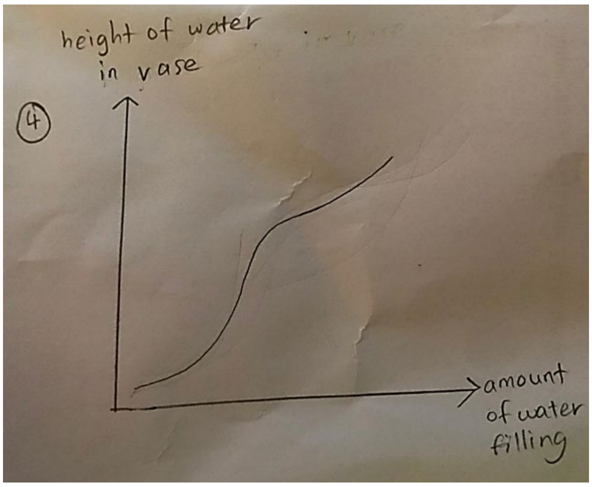

Figure 2

However, her elaboration was contradicted with her graph as she said that the water height increased at slower rate until it reaches middle, faster rate in the middle, and slower rate again until it reaches the top.

“at first, water will rise a bit slowly, I mean like constant rate so the height of … no, no, the water is filling up at a constant rate, so at bottom, the base was huge, so it will rise slow a bit, I mean the height will increase a bit slow, but when it reach middle part, the height will increase a bit fast because the water filling at constant rate, just that the space given is less, I mean the middle part is a bit small so it will rise a bit fast a high rate, the height will increase a bit fast, then the rate will become slow again when the base become wider the top become wider”.

Her sketched graph instead shows a fast rate curved followed by slow rate curved and fast rate curved again. This shows that she conceived concave upwards curved as slower rate, whereas concave downward curved as faster rate. This was clarified as researcher probed her on slope of her graph.

“Ok, first part is a bit slower rate and then middle part faster rate and then back to slower rate”.

“The slope is a bit aa what a bit straight mean more or less like straight mean, it a bit faster or slope is like a bit closer lying down, that is a bit slower rate”.

Despite that, she was able to correctly explain the increase in water height at a slower rate at top part, but she explained that water height increased at a slower rate up until it reached middle part when it supposed to be faster rate. Her inconsistency in her elaboration shows that she is unable to distinguish the change of amount water height correctly for different parts of the vase, hence, stating incorrect changing rate and her sketched graph also were drawn oppositely with her elaboration. Plus, her graph was divided into three parts where it is supposed to be two parts.

On the other hand, Mariam drew an increasing straight-line time-based graph as shown in Figure 3 below and gave the reason due to the flow of water that fills in the vase at a constant rate as mentioned by the question.

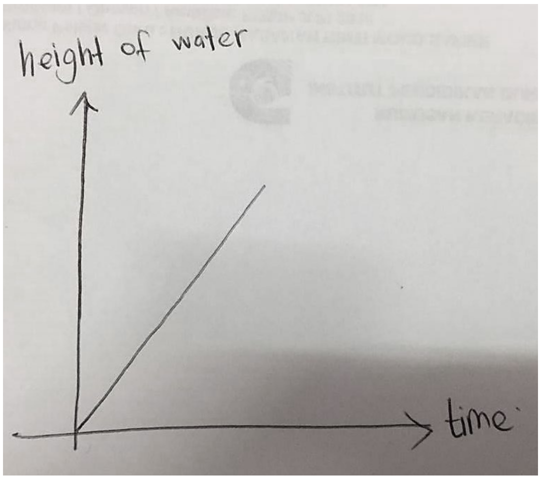

Figure 3

“so, I sketch that graph, for y-axis I put height of water with x-axis I put time, then I draw a graph, it just increasing like the straight line because at the beginning of time like zero time, height of water is minimum then, after that, at the end of time, height of water at the maximum level because its already filled up the vase”.

“Yes, constantly increasing, because if I am not mistaken, the question tells time constant or something like that”

It shows that she is unable to coordinate between two variables of height of water with amount of water filled-in because she appeared to conceive time as an independent variable only instead of amount of water filled-in. Her straight-line graph also shows that she is unable to covariate the changes between independent variable and dependent variable correctly. Although she appeared to have image of coordinating increasing of output with increasing of input (MA2), but her action and solution remarked her inability to conceive amount of water being filled as an independent variable.

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Meanwhile, Johan interchanged between roles of independent and dependent variable, as shown by his graph in Figure 4 below.

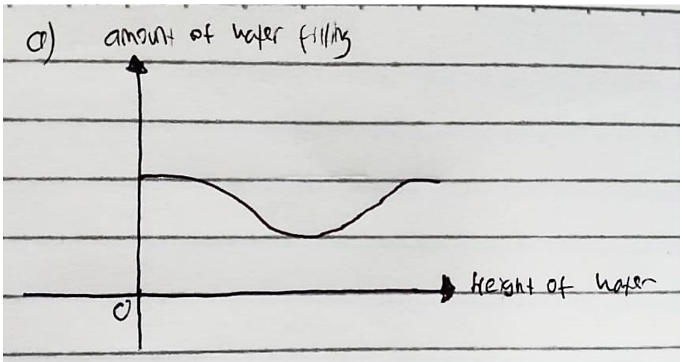

Figure 4

In his explanation, he focused on how the amount of water filled-in changes based on the shape of vase.

"I draw like that because at first, if you see the vase, its shape at first, the wide surface area, then its volume decreasing means at first water needs to enter, amount of water filling in at first is a lot, then it decreases then it becomes greater back, I means the water needed, because it follows shape of vase".

He continued by stating that the volume or amount of water filling-in changes either increases or decreases based on the different surface area of vase curve.

"at first have to fill in a lot, because of the shape, surface area start from bottom is greater, then at middle the surface area become smaller, when becomes smaller, the water needed to fill in is less, but when reach at top, the area become bigger so water needed also become greater".

It was noticed that Johan did not quantified height of water in vase in his elaboration. Apparently, he was seen to mention merely on relationship between amount of water and the surface area of the vase, though his sketched graph included height of water as x-axis.

DISCUSSION

Based on the results, the pre-service mathematics teachers show different kinds of covariation in their reasoning. In term of variables for their covariation, only Arya and Chen are able to quantify the correct variables in their coordination, while Mariam and Johan could not. Arya was able to demonstrate both variables, independent and dependent as she claimed how the water's height changed successively with respect to constant amount of water filled-in, without mentioning any other caused variable, as well as in her labelled graph. Similarly, in a study by Kertil et al. (2019), they classified this as thinking by primary variables where the students are able to conceive the relation between dependent and independent variables, while Carlson et al. (2002) categorized this as the first and foremost mental action in their covariation framework.

Similarly, Chen also demonstrated the relationship between both variables, water's height and amount of water filled-in in his elaboration, except that he also included time as another independent variable in his elaboration and his graph labelled time as x-axis. It seemed that Chen conceived requirement of time as an independent variable, but he also tends to consider amount of water as independent variable as how he emphasized in his elaboration. This indicates that pre-service mathematics teachers still considered the requirement of time as a part of independent variable that must be included while reasoning.

Moreover, Mariam also correspondingly positioned time explicitly as an independent variable both in her elaboration and graph axis, hence, indicated a requirement of time as an independent variable. She did not verbalize the amount of water filled-in unlike Chen, since she only recognized the needs of time as the only independent variable. This was equivalent with findings from Kertil et al. (2019) as they categorized this as thinking by secondary variables, since the students conceived parameter time as a common input in reasoning their covariation. This also aligned with many studies by other researchers (e.g. Carlson & Thompson, 2017;

Herbert & Pierce, 2012; Kertil et al., 2019; Paoletti & Moore, 2017; Şen Zeytun et al., 2010; Yemen et al., 2017), implying how this time-based reasoning was familiarized among pre-service mathematics teachers and students, hence, showing their covariation knowledge must be based by parameter of time. It may also be due to their mindset favouring time as input, which helped them to reduce their cognitive burden as time was easier to act as input, rather than another parameter (Carlson et al., 2002). However, Kertil et al. (2019)'s study shows that students may develop non-temporal time-based as their covariation skills throughout several non-temporal task, as they were guided throughout the model development sequence to conceive how time was not the only utmost input in quantifying relationship between two variables.

Despite that, other cases such as Johan who interchanged between the role of independent and dependent variables are also resembled with previous studiessuch as the studies of Kertil et al., (2019); Paoletti and Moore, (2017); and Yemen et al., (2017). In this situation, the student, apparently, is unable to imagine of how two variables influence each other as they had difficulty to identify between varying variable and constant variable. Hence, these problems emerged from this current study, in determining variables, was categorized under difficulty in quantifying the correct variables to covariate.

In terms of their coordination, Mariam and Johan did not display any coordination. Maryam coordinated height of water increased constantly with respect to time, while Johan coordinated amount of water with respect to height of water. Both of them were unable to determine correct variables for both independent and dependent variables, hence, not demonstrating any coordination between height of water with respect to amount of water. Mariam and Johan, instead, recognized the water was going to increase until it fulfilled the vase. This was similar with no coordination level by Thompson and Carlson (2017), as a result from their study, where students who seemed to acknowledge that water was going up in the bottle, did not make any attempt to coordinate it with amount of water poured into bottle.

As for Chen, he displayed an indirect coordination as he appeared to coordinate the amount of water filling-in indirectly since he also included the variable of time in both his elaboration and graph axis. His explanation, apparently, shows the coordination between height of water with amount of water filling-in, hence are able to coordinate the amount of water height that changed successively with respect to the amount of water filled-in. Similarly, Kertil et al. (2019) categorized this kind of coordination as direct and systematic coordination, while Thompson and Carlson (2017) categorized it as a part of chunky and smooth coordination. In spite of that, Arya also attempted to coordinate the amount of water height changed with respect to amount of water, but she verbalized the changes in amount of height of water incorrectly for the top part of vase, though she is able to coordinate the variables at the bottom part of the vase. Although Arya was able to conceive the relationship between variables, her verbal inconsistency indicated that she was unable to coordinate the amount of water height at different levels of the vase, hence, was unable to observe simultaneous variation in the output variable. Based on Thompson and Carlson (2017), she was considered at a level of gross coordination since her covariation was limited to only conceiving water's height increases as amount of water increases. These problems in coordination between variables may emerge and are related with the difficulty student had in quantitative reasoning. Quantitative reasoning had been tightly related with covariation as mentioned by other researchers (Johnson, 2012; Mkhatshwa, 2020). The reason was due to the emphasis given by Thompson (1993) as he stated that quantitative reasoning was rather a relationship between two or more quantities, than assigning numeric measure to quantity. Hence, as students efficiently engage in quantitative reasoning, he/she would be able to coordinate or compare how two quantities make sense of each other, thus, are able to consider how they change simultaneously, leading to smooth and efficient covariational reasoning (Johnson, 2012).

Consequently, in terms of covariation, only Chen was able to provide a smooth covariation as he stated that if water moves in constantly, rate of height as well as speed of height was increased first and then decreased after passing through the middle part. He gave that the reason was due to the shape of the vase at the middle and for example, in the first half, it narrowed down causing the water that flows in to constantly fill up the area, hence causing the rate of height to fill in increased rapidly. At this level, Chen was said to achieve smooth continuous variation (Thompson & Carlson, 2017) as he was able to anticipate variable changing rate efficiently in each varied-sized interval as shown in the Vase Task. However, as for Arya, she demonstrated the water height increased at slower rate until it reaches middle, faster rate in the middle, and slower rate again until reaches at the top. In her elaboration, she was able to correctly explain water height increased in slower rate at top part, but she also explained water height increased in slower rate up until it reached middle part, where it is supposed to be faster rate. Plus, she elaborated that there were three different changing rates happened, instead of two changing rates. Arya's inconsistency in elaborating the changing rate implies that she was unable to covariate between two quantities efficiently. Hence, this suggests that the correct variables involved could also result in uncoordinated covariation and hence, led to gross coordination. This finding was aligned with a study by Kertil et al. (2019) as they found that students with coordinated thinking could achieve either gross coordination or smooth continuous coordination.

CONCLUSION

The findings of the present study indicated students' difficulties dealing between changes quantities which happened in various stages of covariation. Even at the first stage, to identify variables, the pre-service mathematics teachers were unable to determine the correct variables involved, where they were seen to over attach with time-based parameter and reversing roles between input and output variables. At the second stage of coordination between variables, only Chen was able to coordinate changes in amount of height with respect to amount of water correctly, whereas Arya was observed to coordinate between variables incorrectly at the top part of the vase. To simplify, Chen able to observe changing rate between variables correctly while Arya was not. Hence, these findings indicated that pre-service mathematics teachers' covariation reasoning skills were insufficient. Since this study was limited to solving only one problem and bounded to particularly pre-service mathematics teachers, thus, the researcher did not make any claim to generalize it to other Malaysian pre-service mathematics teachers. This study, instead, gave a picture on difficulties or weaknesses that were found lacking in these future teachers' knowledge, which may help to inform other Malaysian educators or researchers about the enhancement of knowledge, in terms of calculus area. However, the results of this study suggested for teacher training programs to provide a necessary approach or activities to promote these future teachers' covariation skills.

These problems can be encountered by taking a prevention measure in pre-calculus topic, which may help to build up more understanding in rate of change. For instance, student familiarity with time-based parameter may result from an abundant of motion context learning or temporal-based application while attending about rate. Plus, a relationship between two quantities was also first learnt during ratio topic which may help students to engage with multiplicative comparison, hence was able to adopt and conceive how rate of change happened among quantities. These prevention measures may help pre-service teachers to have basic skills in covariation, but providing an adequate learning opportunity is also important. For instance, incorporating real-life covariation task by implementing various activities as shown in the studies by Kertil et al. (2019) and Lynn et al. (2017), has been proven to improve students' covariation skill, hence, foster a better understanding of how rate of change works.

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APPENDICES

Appendix 1 (Vase Task)

Imagine water fills up the given vase below at a constant rate until its fully filled. Hence, based on this situation, sketch a graph of height of water in the vase as a function of amount of water filling up the vase.