COMPARISONS OF HEINZ OPERATOR MEANS WITH DIFFERENT PARAMETERS

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ABSTRACT   This article aims to present new comparisons of Heinz operator means with different parameters by the help of appropriate scalar comparisons and the monotonicity principle for bounded self-adjoint Hilbert space operators. In particular, for any positive operators $A, B \in B(H)$, we establish the inequality

$$H_\tau(A,B) \leq \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)(A\nabla B) + \frac{(1-2\mu)^2}{(1-2\tau)^2} H_\mu(A,B),$$

where $\mu, \tau \in [0,1]$ satisfying

$$\tau \neq \frac{1}{2}, \left|\tau - \frac{1}{2}\right| \geq \left|\mu - \frac{1}{2}\right| \text{ and } |2\tau - 1| + |2\mu - 1| \leq 1.$$

INTRODUCTION

The Heinz mean of positive numbers a, b in the parameter $v \in [0,1]$, defined by

$$H_v(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}$$

ICMSS2018
3rd International Conference of Mathematical Sciences and Statistics
6th - 8th February 2018

is one of the means which intermediates between the arithmetic mean and geometric mean, that is

$$\sqrt{ab} \leq H_v(a,b) \leq \frac{a+b}{2}, \qquad 0 \leq v \leq 1 \tag{1}$$

The function $H_v(a,b)$, is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$. Thus,

$$H_\mu(a,b) \leq H_\tau(a,b), \qquad 0 \leq \tau \leq \mu \leq \frac{1}{2} \tag{2}$$

and

$$H_\mu(a,b) \leq H_\tau(a,b), \qquad \frac{1}{2} \leq \mu \leq \tau \leq 1 \tag{3}$$

More interesting inequalities related to the Heinz means can be found in (Bhatia, 2006). An operator version of (1) due to (Bhatia and Davis, 1993) is the following inequalities

$$\left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\| \leq \left\| \frac{A^v B^{1-v} + A^{1-v} B^v}{2} \right\| \leq \left\| \frac{A+B}{2} \right\|, \qquad 0 \leq v \leq 1, \tag{4}$$

where A, B are positive operators on a complex separable Hilbert space and $\|\cdot\|$ is a unitarily invariant norm. Usually, $\left\| \frac{A^v B^{1-v} + A^{1-v} B^v}{2} \right\|$ is called the Heinz mean of A and B. Several inequalities improving the inequalities in (4) have been given by (Kittaneh, 2010; Kittaneh & Manasrah, 2010; Zhan, 1998).

Throughout this article, the space of all bounded linear operators on a Hilbert space H will be denoted by $B(H)$. In the case when $\dim H = n$, $B(H)$ is identified with the matrix algebra M_n of all $n \times n$ matrices with complex entries.

In the following, another operator version of Heinz mean is considered. By recognizing the definition in the scalar case, the Heinz mean is the arithmetic mean of two weighted geometric means. Hence, such definition can be raised up to the level of operators via the operator means. Following (Kittaneh et al., 2012), the weighted arithmetic operator mean ∇_v and geometric mean $\#_v$ are defined as follows:

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$$A\nabla_v B = (1 - v)A + vB,$$

$$A\#_v B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^v A^{\frac{1}{2}},$$

for $v \in [0,1]$ and positive invertible operators $A, B \in B(H)$.

If $v = \frac{1}{2}$, we write $A\nabla B$ and $A\#B$ to denote the arithmetic operator mean and geometric operator mean, respectively.

By regarding the above definitions, the Heinz operator mean is defined as

$$H_v(A, B) = \frac{A\#_v B + A\#_{1-v}B}{2}.$$

It is well known that the Heinz operator mean intermediates between the arithmetic operator mean and geometric operator mean, that is

$$A\#B \le H_v(A, B) \le A\nabla B. \tag{5}$$

In recent years such operator means and related comparisons have been under active investigations. The authors in (Kittaneh et al., 2012) established the following improvement of the second inequality in equation (5) above

$$2min\{v, 1 - v\}(A\nabla B - A\#B) \le A\nabla B - H_v(A,B).$$

The same result was established, in (Kittaneh & Manasrah, 2011), for matrices. Moreover, Kittaneh and Krnić (2013), derived a series of improvements of the Heinz operator inequality by using the *Hermite - Hadamard inequality*.

$$A\#B \le H_{\frac{2v+1}{4}}(A,B) \le \frac{1}{4}H_v(A,B) + \frac{1}{2}H_{\frac{2v+1}{4}}(A,B) + \frac{1}{4}A\#B$$

$$\le \frac{1}{2}H_v(A,B) + \frac{1}{2}A\#B \le H_v(A,B).$$

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In this paper, we are concerned with finding new ordering relations between the Heinz operator means with different parameters by the help of appropriate scalar inequalities and the monotonicity principle for bounded self-adjoint operators on Hilbert space (see, e.g. Pečarić et al., 2005): Let $X \in B(H)$ be self-adjoint with a spectrum $Sp(X)$ and let g and h be continuous real functions such that $g(x) \geq h(x)$ for all $x \in Sp(X)$. Then $g(X) \geq h(X)$.

MAIN RESULTS

The main objective of this section is to derive new comparisons of Heinz means with different parameters for scalars and operators.

Theorem 2.1. *If $a, b > 0$ and $\mu, \tau \in [0,1]$ satisfying*

$$\tau \neq \frac{1}{2}, \left|\tau - \frac{1}{2}\right| \geq \left|\mu - \frac{1}{2}\right| \text{ and } |2\tau - 1| + |2\mu - 1| \leq 1,$$

then

$$H_\tau(a,b) \leq \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{a+b}{2}\right) + \frac{(1-2\mu)^2}{(1-2\tau)^2} H_\mu(a,b). \tag{6}$$

Proof: Let $\rho = 1 - 2\tau$ and $\omega = 1 - 2\mu$. Then $\rho \neq 0, 0 \leq |\omega| \leq |\rho| \leq 1$.

By using the Taylor expansion of $\cosh x,$ we have

$$\cosh \rho x = 1 + \frac{\rho^2 x^2}{2!} + \frac{\rho^4 x^4}{4!} + \cdots$$

$$\leq 1 + \left(1 - \frac{\omega^2}{\rho^2} + \frac{\omega^4}{\rho^4}\right)\frac{x^2}{2!} + \left(1 - \frac{\omega^2}{\rho^2} + \frac{\omega^6}{\rho^2}\right)\frac{x^4}{4!} + \cdots$$

$$= \left(1 - \frac{\omega^2}{\rho^2}\right)\cosh x + \frac{\omega^2}{\rho^2}\cosh \omega x.$$

Replacing x by $\frac{x-y}{2}$, we have

$$\cosh\left((1-2\tau)(\frac{x-y}{2})\right)$$

$$\leq \left(1-\frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\cosh\left(\frac{x-y}{2}\right)+\frac{(1-2\mu)^2}{(1-2\tau)^2}\cosh\left((1-2\mu)(\frac{x-y}{2})\right).$$

Put $a=e^x$ and $b=e^y$, to get

$$\frac{a^\tau b^{1-\tau}+a^{1-\tau}b^\tau}{2}\leq\left(1-\frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{a+b}{2}\right)+\frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{a^\mu b^{1-\mu}+a^{1-\mu}b^\mu}{2}\right).$$

This completes the proof.

In view of Theorem 2.1, we have

- if $0\leq\tau\leq\mu\leq\frac{1}{2}$, $\tau\neq\frac{1}{2}$ and $|2\tau-1|+|2\mu-1|\leq 1$, then

$$H_\tau(a,b)\leq\left(1-\frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{a+b}{2}\right)+\frac{(1-2\mu)^2}{(1-2\tau)^2}H_\mu(a,b)$$

- if $\frac{1}{2}\leq\mu\leq\tau\leq 1$, $\tau\neq\frac{1}{2}$ and $|2\tau-1|+|2\mu-1|\leq 1$, then

$$H_\tau(a,b)\leq\left(1-\frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{a+b}{2}\right)+\frac{(1-2\mu)^2}{(1-2\tau)^2}H_\mu(a,b),$$

for $a,b>0$.

Thus, our result in Theorem 2.1 is a reverse of inequalities in (2) and (3) under the given conditions.

Now, by virtue of the monotonicity principle and based on the inequality (6), we obtain our first comparison of Heinz operator means.

Theorem2.2
If $A,B\in$
$B(H)$ are positive definite and $\mu,\tau\in$
$[0,1]$ satisfying

$$\tau\neq\frac{1}{2},\ \left|\tau-\frac{1}{2}\right|\geq\left|\mu-\frac{1}{2}\right|\ \text{and}\ |2\tau-1|+|2\mu-1|\leq 1,$$

then

$$H_\tau(A,B) \le \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)(A\nabla B) + \frac{(1-2\mu)^2}{(1-2\tau)^2}H_\mu(A,B). \tag{7}$$

Proof: Let $X = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. Then for any $x \in Sp(X)$, we have

$$\frac{x^\tau + x^{1-\tau}}{2} \le \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{x+1}{2}\right) + \frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{x^\mu + x^{1-\mu}}{2}\right).$$

Now, by the monotonicity principle, we have

$$\frac{X^\tau + X^{1-\tau}}{2} \le \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{X+I}{2}\right) + \frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{X^\mu + X^{1-\mu}}{2}\right).$$

Replacing X by $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and multiplying both sides by $A^{\frac{1}{2}}$, we obtain

$$H_\tau(A,B) \le \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)(A\nabla B) + \frac{(1-2\mu)^2}{(1-2\tau)^2}H_\mu(A,B).$$

This completes the proof.

In the following, we establish a refinement of the second inequality in (4) for the Hilbert-Schmidt norm under given conditions.

Theorem 2.3. *If $A, B, X \in M_n$, where A and B are positive definite and $\mu, \tau \in [0,1]$*

satisfying

$$\tau \ne \frac{1}{2}, \quad \left|\tau - \frac{1}{2}\right| \ge \left|\mu - \frac{1}{2}\right| \text{ and } |2\tau - 1| + |2\mu - 1| \le 1,$$

then

$$\left\| \frac{A^\tau X B^{1-\tau} + A^{1-\tau} X B^\tau}{2} \right\|_h$$

$$\leq \left\| \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{AX+XB}{2}\right) + \frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu}{2}\right) \right\|_h.$$

Proof: Every positive definite matrix is unitarily diagonalizable and this implies that there are unitary matrices $U, V \in M_n$ such that $A = U D_1 U^*$ and $B = V D_2 V^*$, where $D_1 = diag(\gamma_1, \dots, \gamma_n)$ and $D_2 = diag(\sigma_1, \dots, \sigma_n)$ with $\gamma_i, \sigma_i > 0$, $i = 1, \dots, n$. Let $Y = U^* X V = [y_{ij}]$. Then, we get $AX + XB = U[(\gamma_i + \sigma_j)y_{ij}]V^*$, and $A^\tau X B^{1-\tau} + A^{1-\tau} X B^\tau = U[(\gamma_i^\tau \sigma_j^{1-\tau} + \gamma_i^{1-\tau}\sigma_j^\tau)y_{ij}]V^*.$

Thus,

$$\left\| \frac{A^\tau X B^{1-\tau} + A^{1-\tau} X B^\tau}{2} \right\|_h^2 = \sum_{i,j=1}^n \left(\frac{\gamma_i^\tau \sigma_j^{1-\tau} + \gamma_i^{1-\tau}\sigma_j^\tau}{2}\right)^2 |y_{ij}|^2$$

$$\leq \sum_{i,j=1}^n \left(\left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{\gamma_i + \sigma_j}{2}\right) \right.$$

$$\left. + \frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{\gamma_i^\mu \sigma_j^{1-\mu} + \gamma_i^{1-\mu}\sigma_j^\mu}{2}\right) \right)^2 |y_{ij}|^2$$

$$= \left\| \left(1 - \frac{(1-2\mu)^2}{(1-2\tau)^2}\right)\left(\frac{AX+XB}{2}\right) \right.$$

$$\left. + \frac{(1-2\mu)^2}{(1-2\tau)^2}\left(\frac{A^\mu X B^{1-\mu} + A^{1-\mu} X B^\mu}{2}\right) \right\|_h^2.$$

This completes the proof.

In what follows, we turn to another mean that intermediates between the geometric mean and arithmetic mean, which is the logarithmic mean. The logarithmic mean is defined as

$$L(a, b) = \frac{\log a - \log b}{a - b},$$

where, $a, b > 0$ and

$$\sqrt{ab} \leq L(a, b) \leq \frac{a + b}{2}. \tag{8}$$

In order to establish a series of refinements of Heinz inequalities, Kittaneh and Krnić, (2013) considered the parameterized class of continuous functions $F_v: R^+ \to R, v \in [0,1],$ defined by

$$F_v(x) = \begin{cases} \dfrac{x^v - x^{1-v}}{\log x}, & x > 0, x \neq 1, \\ 2v - 1, & x = 1. \end{cases}$$

In the following theorem, we construct another refinement of Heinz inequalities, which is a generalization of the second inequality in (8).

Theorem 2.4. *If* $a, b > 0$, $\frac{1}{3} \leq k \leq 1$ *and* $\mu, \tau \in [0,1]$ *satisfying*

$$\tau \neq \frac{1}{2}, \quad \left|\tau - \frac{1}{2}\right| \geq \left|\mu - \frac{1}{2}\right| \text{ and } |2\tau - 1| + |2\mu - 1| \leq 1,$$

then

$$\frac{a^{1-\tau}b^\tau - a^\tau b^{1-\tau}}{(1 - 2\tau)(\log a - \log b)} \leq \left(1 - \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}k\right)\left(\frac{a + b}{2}\right) + \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}kH_\mu(a, b). \tag{9}$$

Proof: Let $\rho = 1 - 2\tau$ and $\omega = 1 - 2\mu$. Then $\rho \neq 0, 0 \leq |\omega| \leq |\rho| \leq 1$.

By using the Taylor expansion of $\sinh x$ and $\cosh x$, we have

$$\frac{\sinh \rho x}{\rho x} = 1 + \frac{\rho^2 x^2}{3!} + \frac{\rho^4 x^4}{5!} + \cdots$$

$$\leq 1 + \left(1 - \frac{\omega^2}{\rho^2} + \frac{\omega^4}{\rho^2}\right)\frac{x^2}{3!} + \left(1 - \frac{\omega^2}{\rho^2} + \frac{\omega^6}{\rho^2}\right)\frac{x^4}{5!} + \cdots$$

$$\leq 1 + \left(\frac{1}{k} - \frac{\omega^2}{\rho^2} + \frac{\omega^4}{\rho^2}\right)\frac{x^2}{3!} + \left(\frac{1}{k} - \frac{\omega^2}{\rho^2} + \frac{\omega^6}{\rho^2}\right)\frac{x^4}{5!} + \cdots$$

$$\leq 1 + k\left(\frac{1}{k} - \frac{\omega^2}{\rho^2} + \frac{\omega^4}{\rho^2}\right)\frac{x^2}{2!} + k\left(\frac{1}{k} - \frac{\omega^2}{\rho^2} + \frac{\omega^6}{\rho^2}\right)\frac{x^4}{4!} + \cdots$$

$$= 1 + \left(1 - \frac{\omega^2 k}{\rho^2} + \frac{\omega^4 k}{\rho^2}\right)\frac{x^2}{2!} + \left(1 - \frac{\omega^2 k}{\rho^2} + \frac{\omega^6 k}{\rho^2}\right)\frac{x^4}{4!} + \cdots$$

$$= \left(1 - \frac{\omega^2}{\rho^2}k\right)\cosh x + \frac{\omega^2}{\rho^2}k \cosh \omega x.$$

Replacing x by $\frac{x-y}{2}$, we have

$$\frac{\sinh\left((1 - 2\tau)\left(\frac{x-y}{2}\right)\right)}{(1 - 2\tau)\left(\frac{x-y}{2}\right)}$$

$$\leq \left(1 - \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}k\right)\cosh\left(\frac{x-y}{2}\right) + \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}k \cosh\left((1 - 2\mu)(\frac{x-y}{2})\right).$$

Put $a = e^x$ and $b = e^y$, to get

$$\frac{a^{1-\tau}b^\tau - a^\tau b^{1-\tau}}{(1 - 2\tau)(\log a - \log b)} \leq \left(1 - \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}k\right)\left(\frac{a+b}{2}\right) + \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2}k\left(\frac{a^{1-\mu}b^\mu + a^\mu b^{1-\mu}}{2}\right).$$

This completes the proof.

Taking $\tau = 1$ or 0 in Theorem 2.4, this yields $\mu = \frac{1}{2}$ and so the second inequality in (8) is obtained.

Finally, an operator version of the inequality (9) is obtained by using the monotonicity principle in a similar way as in the proof of Theorem 2.2.

Theorem 2.5. *If $A, B \in B(H)$ are positive definite, $\frac{1}{3} \le k \le 1$ and $\mu, \tau \in [0,1]$ satisfying*

$$\tau \ne \frac{1}{2}, \ \left|\tau - \frac{1}{2}\right| \ge \left|\mu - \frac{1}{2}\right| \text{ and } |2\tau - 1| + |2\mu - 1| \le 1,$$

then

$$\frac{1}{(2\tau - 1)} A^{\frac{1}{2}} F_\tau(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \le \left(1 - \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2} k\right)(A\nabla B) + \frac{(1 - 2\mu)^2}{(1 - 2\tau)^2} k \, H_\mu(A, B).$$

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