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THE BOOK AL-JABR WA AL-MUQĀBALAH: A RESEARCH ON ITS CONTENT, WRITING METHODOLOGY AND ELEMENTARY ALGEBRA BY AL-KHWĀRIZMĪ

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Khulasah

Islam suatu ketika dahulu pernah mencapai zaman kegemilangan tamadunnya. Banyak karya-karya yang ditulis hasil dari penguasaan pelbagai bidang ilmu oleh tokoh-tokoh kesarjanaan Islam. Kajian ini akan membincangkan sebuah hasil karya seorang tokoh tersohor, al-Khwārizmī melalui bukunya al-Jabr wa al-Muqābalah. Kertas kerja ini akan cuba memperlihatkan pembahagian bab-bab kandungan kitab ini, gaya penulisannya serta kedudukan kitab ini dari sudut sumbernya. Kajian ini juga difokuskan kepada asas algebra yang dikemukakan oleh beliau dalam bahagian pertama kitab ini. Ia merangkumi istilah-istilah matematik yang kerap digunakan di dalam kitab ini, contoh-contoh permasalahan yang terlibat dan juga cara penyelesaian yang ditunjukkan oleh al-Khwārizmī terhadap masalah-masalah tersebut. Ia diharap dapat memberi peluang kepada pembaca khususnya di Malaysia untuk mempelajari sesuatu hasil dari sejarah ketamadunan Islam, menjadi nilai tambah kepada sumber dan bahan yang sedia ada, dan seterusnya memudahkan pembaca untuk memahami pokok perbincangan kitab ini untuk dibincangkan dalam konteks yang lebih meluas.

Kata Kunci: al-Khwārizmī, *al-Jabr wa al-Muqābalah,* algebra, persamaan kuadratik.

Abstract

The golden age of Islamic civilization reached its peak between 8th century and 13th century. Many treatises were written as a result of studies in various fields by Islamic scholars. This article will discuss a masterpiece by a grand scholar, al-Khwārizmī through his book al-Jabr wa al-Muqābalah. This article will show the divisions of content in the book, its writing style as well as sources of the book. The study also will be focusing on basic algebra introduced by him in the first part of the book. It covers mathematical terms frequently used in this book, the examples of the mathematical problems involved and their solutions shown by al-Khwārizmī. It is hopeful that this article will provide an opportunity to the readers, especially in Malaysia to learn something as a result of the history of Islamic civilization, will give additional value to the resources and materials available, thus facilitating the readers to understand the debates in this book to be discussed in a broader context.

Keywords: al-Khwārizmī, *al-Jabr wa al-Muqābalah,* algebra, quadratic equations.

Introduction

Mathematics is directly connected to the essence of Islamic message, which is the principle of Unity (*al-Tawhīd*). Allah is one, hence the number one in the series of numbers is the most direct and most understandable symbol.[1] The Muslims are able to understand what have

[1] Seyyed Hossein Nasr, *Islamic Science: An Illustrated Study* (England: World of Islam Festival Publishing Company Ltd., 1976), 75.

been called upon by the Prophet to believe the oneness of Allah and His creation. They also need mathematics to establish religious commands such as the amount of *rakā'ah* in prayers, the total number of the days of fasting during the holy month of Ramaḍan, the *Zakāh* measurement and the *Ḥājj*.[2] The Muslims also needed mathematics to determine the direction of *Qiblah*.[3]

In the Qur'an, there are a lot of verses mentioning numbers. For instance, in Sūrah al-Nisā' (4) verse 10, 11 and 12 describe the inheritance in Islam by giving two-third, one-halve, one-third, one-fourth, one-sixth and one-eighth as examples.[4]

The Muslim calendar begins in the year 622 A.D, when Prophet Muhammad fled from his hometown of Mecca, to Medina for eight years.[5] This flight, known as the Hegira, marked the beginning of the Islamic calendar, one that was exert a strong influence on the development of mathematics.[6]

During the life of Prophet Muhammad p.b.u.h, Arabic letters which are known as *Hijā'iyyah* have been used as numbers as shown in Figure 1.1.[7] They used letters to symbolize numbers.[8] Every letter represented number 1 to 10, 20 to 90, followed by hundreds and

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[2] *Tamadun Islam dan Tamadun Asia* (Kuala Lumpur: Universiti Malaya, 2004), 106.

[3] Ali Abdullah al-Daffa', *Sumbangan Islam Dalam Bidang Matematik,* transl. Amin Senin (Kuala Lumpur: Dewan Bahasa dan Pustaka, 1988), 1.

[4] Amir A. Rahman, *Pengantar Tamadun Islam* (Kuala Lumpur: Dewan Bahasa dan Pustaka, 1991), 119.

[5] J. L. Berggren, *Episodes in the Mathematics of Medieval Islam* (New York: Springer-Verlag, 1986), 1.

[6] C. B. Boyer, *A History of Mathematics*, 2nd Edition (New York: John Wiley & Sons, Inc., 1991), 225.

[7] Ali Abdullah al-Daffa', *Sumbangan Islam Dalam Bidang Matematik,* 29.

[8] Mahmood Zuhdi Ab. Majid, *Tokoh-Tokoh Kesarjanaan Sains Islam* (Kuala Lumpur: Dewan Bahasa dan Pustaka, 2003), 81.

thousands. All of these symbols were used by the early Arabs to do calculation of addition, subtraction, multiplication and division.[9] This method is known as *hisāb al-Jummāl*.

1	ا	10	ي	100	ق	1000	غ	10000	يغ	100000	قغ
2	ب	20	ك	200	ر	2000	بغ	20000	كغ	200000	رغ
3	ج	30	ل	300	ش	3000	جغ	30000	لغ	300000	شغ
4	د	40	م	400	ت	4000	دغ	40000	مغ	400000	تغ
5	ه	50	ن	500	ث	5000	هغ	50000	نغ	500000	ثغ
6	و	60	س	600	خ	60000	وغ	60000	سغ	600000	خغ
7	ز	70	ع	700	ذ	7000	زغ	70000	عغ	700000	ذغ
8	ح	80	ف	800	ض	8000	حغ	80000	فغ	800000	ضغ
9	ط	90	ص	900	ظ	90000	طغ	90000	صغ	900000	ظغ

Figure 1.1: The Early Arabs used letters to represent numbers.

For example, if 20 + 30 = 70, the Arabs will write it this way:

ن = لْ + ک

During the reign of Caliph 'Umar al-Khaṭṭāb (634-644 A.D), a National Treasury Department was established to manage the revenue of the nation systematically for the welfare of the Muslims; this became known later as Bayt al- Māl.[10]

We faced everywhere in the world things related with mathematics. It is because of this element within the total spectrum of Islamic spirituality that Muslims became attracted to the various branches of mathematics early in

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[9] Sulayman Fayyad, *Al-Khwarizmi: Ahli Matematik,* transl. Abdul Karim Mustafa (Kuala Lumpur: Dewan Bahasa dan Pustaka, 2003), 26.

[10] Talib Samat, *Umar Al-Khattab Khalifah Al-Rasyidin Kedua* (Shah Alam: Marwilis Publisher & Distributors Sdn. Bhd., 1988), 143.

their history and made so many contributions to the mathematical sciences for nearly a millennium.[11]

The work of translations by Muslims and non-Muslims scholars had been known for centuries. These activities can be seen in the Islamic world during the period of Abbasid Caliphs, where the works of translations were rapidly and actively developed. The encouragement by the Abbasid rulers especially Hārūn al-Rashīd (766-809 A.D) and Caliph al-Makmūn (813-833 A.D) during these periods have successfully brought about a number of scholars in various fields and disciplines including al-Khwārizmī.

This article will be focusing on al-Khwārizmī, a scholar in mathematics, astronomy and geography as the main research topic, particularly in the field of algebra founded by him. The debate in this article will be on his main treatise, *al-Jabr wa al-Muqābalah* and elementary algebra introduced by him. Indeed, al-Khwārizmī was chosen for this research is because of his significant contribution as the first person to invent algebra not only in the Islamic world but also the world at large. The main objective al-Khwārizmī wrote the book *al-Jabr wa al-Muqābalah* is to help the Muslims to solve cases of inheritance (*farā'iḍ*), partition, law-suits, general computation in business and trade.[12]

It would be hopeful that this research will be very useful to broaden the literatures for further use by scholars, scientists and students. Important values will be reflected from the contributions of al-Khwārizmī in continuing the intellectual traditions in the history of mathematics, making this noble mathematician lasting to our memory.

[11] Hossein Nasr, *Islamic Science*, 77.

[12] L. C. Karpinski & J. G. Winter, *Contributions to the History of Science* (New York: Johnson, 1972), 46.

Al-Khwārizmī and his Contributions

His full name is Abū ‘Abdullāh Muḥammad ibn Mūsā al-Khwārizmī.[13] His birth and death dates were not accurately known; some said it was between 800-847 A.D. and others said it was earlier between 780-850 A.D. He originated from Khwārizm, located on the south of Aral Sea in Turkistan. He moved to Baghdad and flourished under the reign of Abbasid Caliph al-Ma’mūn (813-833 A.D).[14] It was during this period that the House of Wisdom (Bayt al-Ḥikmah) was established in 815 A.D. Not much biographical information is available about this great mathematician, astronomer and geographer. However, his contributions towards science have made a significant change in the history of science in Islam.[15]

In astronomy, al-Khwārizmī was the first astronomer to interpret and compute the astronomical and trigonometrical tables.[16] His treatise entitled *Zīj al-Sindhind* (*Astronomical Table of Sind and Hind*) was based on the Indian astronomical methods and it was Al-Ma'mūn who appointed him as the royal astronomer. In geography, al-Khwārizmī was the first scholar to introduce and draw the world map.[17] Al-Khwārizmī wrote a geographical treatise, entitled *Ṣūrah al-Arḍ* (*The Face of the Earth*) which was essentially a revised and completed edition of Ptolemy’s geography.[18] It was also called *The Image of the Earth*.[19]

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[13] Mohaini Mohamed, *Great Muslim Mathematicians* (Skudai, Johor: Universiti Teknologi Malaysia, 2000), 17.

[14] G. Sarton, *Introduction to the History of Science* (Baltimore, MD: The Williams & Wilkins Company, 1927), 563.

[15] F. Rosen, *al-Khwarizmi’s Algebra* (Islamabad: Pakistan Hijra Council, 1989).

[16] G. Sarton, *Introduction to the History of Science*, 563.

[17] M. Atiqul Haque, *Muslim Heroes of the World* (Kuala Lumpur: Synergy Books International, n.d), 34.

[18] G. Sarton, *Introduction to the History of Science*, 563.

[19] J. L. Berggren, *Episodes in the Mathematics of Medieval Islam,* 9.

In mathematics, al-Khwārizmī marks the beginning of that period of mathematical history.[20] He wrote more than half a dozen mathematical works.[21] The introduction of decimal positional system that the Hindus had developed by the 6th century was depicted in al-Khwārizmī's arithmetical work in the early 9th century, *al-Jam` wa al-Tafrīq bi Hisāb al-Hind* (*The Book of Addition and Subtraction According to the Hindu Calculation*), which was the oldest known text of such kind.[22] However, the Arabic text seems to be lost but can be reconstructed from the Latin translation made in Spain in the 12th century, entitled *De Numero Indorum*.[23]

In the history of algebra, al-Khwārizmī's most famous book is entitled *al-Jabr wa al-Muqābalah*.[24] It was widely known and may be regarded as the foundation and keystone of algebra.[25] The word algebra derived from the word *al-Jabr*. The word algorithm in English, *guarismo* in Spanish and *algarismo* in Portuguese was originated from his name, which was defined as the art of calculating in any particular way[26].

Al-Khwārizmī is internationally known as the Father of Algebra because of his significant contribution in mathematics.[27] According to George Sarton in his book *Introduction to the History of Science*, in the first half on 9th century he may be regarded as The Greatest Mathematician of The Time, and if one takes all circumstances into account, one of the greatest of all times

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[20] L. C. Karpinski & J. G. Winter, *Contributions to the History of Science*, 13.
[21] C. B. Boyer, *A History of Mathematics,* 227.
[22] Hossein Nasr, *Islamic Science,* 78.
[23] *Ibid.,* 3.
[24] J. L. Berggren, *Episodes in the Mathematics of Medieval Islam,* 7.
[25] Mohaini Mohamed, *Great Muslim Mathematicians,* 18.
[26] *Ibid.,* 18.
[27] Mohammad Yasin Owadally, *The Muslim Scientists*, ed. Abu Tahir (Kuala Lumpur: A.S. Noordeen, 2003), 2.

was al-Khwārizmī.[28] He influenced mathematical thought to a greater extent than any other medieval writer.[29] According to Roshdi Rashed, al-Khwārizmī's work is extensive, covering mathematics including arithmetic and algebra, astronomy and its subsidiary areas, study of instruments, chronology and geography, as well as history.

A comment by Jeffrey A. Oaks is that one way historians have misinterpreted al-Khwārizmī's algebra is that they have presumed that the book is a scientific exposition of algebra. In fact, it is a practical book, and the rules presented in the first part are designed specifically to aid the student is solving the worked-out problems in the second part. The rest of the book contains chapters not normally included in algebra books, mercantile transactions and inheritance, but al-Khwārizmī adds them to show how algebra applies to these areas.

Editions and Translation of the Book

It is known that there were seven existing manuscipts of *al-Jabr wa al-Muqābalah*. The combinations of these manuscripts have been retrieved from several countries and are combined, producing an edition by Roshdi Rashed in 2009 in his latest edition on algebra. For the purpose of this research, the researcher has taken this edition as the main source as it is the latest version to al-Khwārizmī's book.

There is also another earlier edition found by the researcher, written by Frederic Rosen in 1831. If comparison is to be made between the two, Roshdi Rashed has used all the sources including the one kept in Bodleain Library, Oxford, United Kingdom used by Frederic Rosen in his edition. The edition by Roshdi Rashed has also included six other manuscripts that had not been used by Frederic Rosen. There is also no paragraph or chapters in

[28] G. Sarton, *Introduction to the History of Science*, 545.
[29] *Ibid.*, 563.

Rosen's edition, and it is noticed there were words and phrases missing in this version, making the sentences incomplete. However, the version by Roshdi Rashed has structured the chapters and sentences, making it readable and easier to understand.

As previously mentioned, the book *al-Jabr wa al-Muqābalah* has been translated to several languages, including English, French and Latin. Roshdi Rashed translated it to French and published it in 2007. It was followed by an English translation of the book by J.V Field from the French translation by the same author. It has been published in 2009.

For the version by Frederic Rosen, it has also been published together with English translation entitled *The Algebra of Mohammed ben Mūsā*. There were three authors who translated it to Latin in three different versions. They are Gerard of Cremona, Robert of Chester and William of Luna.

The researcher also found that there is another book with the same title, but written by a different author, that is Abū Kāmil Shujā' bin Aslām (850-930 A.D). The treatise is in a form of manuscript, but has been reproduced and edited by Fuat Sezgin in 1986. It has expanded the basic algebra with wider and broader discussion.

There are also commentaries towards al-Khwārizmī's algebra. For instance, by B.L van der Waerden, entitled *A History of Algebra: From al-Khwārizmī to Emmy Noether*. But the researcher has only found commentary by Louis Charles Karpinski and John Garrett Winter regarding the Latin translation by Robert of Chester entitled *Liber Algebrae Et Almucabola*. The book comprises of Hebrew text, translation and commentaries with special reference to the Arabic text.

Content and Structure of the Book

In general, *al-Jabr wa al-Muqābalah* is divided into two major books (*kitāb*). The first book is on the theories of algebra and the second book is on the application of algebra in line to the theories presented. In both books contain chapters and divisions of small chapters. The first book is divided into 8 chapters. Each chapter contains a few small chapters.

The first chapter is the introduction, of which begins with praises be to Allāh SWT, followed by al-Khwārizmī's appreciation and gratitude to the Caliph al-Ma'mūn (786-833 A.D). An interesting phrase expressed by al-Khwārizmī in the beginning of his chapter:

> "…When I see what people generally need in calculating, I found that everything is related to numbers…"[30]

From there, al-Khwārizmī introduced six quadratic equations in the same chapter that form the basis of the first book. The six quadratic equations in the first chapter are divided into two small chapters. They are simple equations and combined trinomials. It is followed by second chapter on multiplication, and then third chapter on addition and subtraction. The fourth chapter is called division and multiplication of roots.

Next, al-Khwārizmī produced six problems related to the six quadratic equations together with their solutions in the fifth chapter, named chapter on the six problems. It is divided into six small chapters, representing six problems consecutively. He went on to produce more examples of mathematical problems with similar solutions in chapter six; chapter on various problems.

The first book continues with the seventh chapter on transactions (*mu`āmalah*), with examples of mathematical

[30] Roshdi Rashed, *al-Khwarizmi: The Beginnings of Algebra* (Beirut: Saqi Publication, 2009), 97.

problems in cases of sale. Then the last chapter concludes the first book with a chapter on measurement on area of square, rectangle, triangle, circle and trapezium.

The second book involves deeper discussions involving the applications in new science, which was created after algebra, which is the science of *farā'iḍ*.[31] Topics discussed in this book involves the division of *farā'iḍ*, debt, inheritance, wills, dowries and giving slaves as hereditary property. The *farā'iḍ* touches the cases of debt, wills, legacies, forms of donation, marriage during illness, dowry and slaves become legacy, capital and profit.

The second book is divided into nine chapters. The first chapter in on asset and debt followed by two chapters namely another chapter on wills. Next, there are three small chapters with same name. They are stated as on another kind of wills.

The fourth chapter is a chapter on the will with a dirham. The fifth chapter is chapter on complement, followed by chapter thereof on marriage during illness. The seventh chapter is chapter on freeing slaves during illness. Then, it is a chapter on the return of the dowry and the last chapter, chapter nine is chapter on forward buying during illness.

Writing Methodology of the Book

If we take a look at the Arabic version of the edition by Roshdi Rashed, it is interesting to observe how in al-Khwārizmī's writing, there are no symbols of numbers used in the sentence structures. All his discussions were explained by words, even though the book's main discussion is about mathematics.

All the mathematical problems displayed by al-Khwārizmī were explained in the form of long sentences,

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[31] Interview with Profesor Roshdi Rashed, on 18 January 2010, Bourg-la-Reine, France.

until he could determine the solution. This is the writing technique used by al-Khwārizmī.

There are mathematical solutions which are illustrated in the form of geometrics or diagrams, but these illustrations were only displayed after al-Khwārizmī had explained those solutions in lengthy sentences. Even the illustrations were marked by letters instead of numbers. Examples on this would be introduced in the next chapter.

The Definitions of al-Jabr wa al-Muqābalah

Traditionally, the word algebra means an understanding of obtaining unknown data given, if there is a relationship between the data.[32] In the modern world, the word algebra brings us to a more complex definition, as this branch of knowledge has expanded through centuries. There are Linear Algebra, Algebraic Equations, Quadratic Forms, and Vector Algebra, to name a few. The ongoing research indicates that the algebra used by al-Khwārizmī belong to both traditions. As for the modern meaning it falls under the category of elementary algebra, of which includes what is now known as Linear and Quadratic Algebra. The best possible way to describe al-Khwārizmī's algebra in modern world is the part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations.[33]

Various definitions from various authors have been given on the meaning of *al-Jabr* and *al-Muqābalah.* Two processes are involved. They are *al-Jabr* (restoring or completion) and *al-Muqābalah* (balancing). *Al-Jabr* is the process of removing negative units, roots and squares from the equation by adding the same quantity to each side.[34] Some said the word *al-Jabr* seems to refer to the

[32] Ibn Khaldun, *Mukadimah Ibn Khaldun,* transl. Dewan Bahasa dan Pustaka (Kuala Lumpur: Dewan Bahasa dan Pustaka, 2006), 667.
[33] Mohaini Mohamed, *Great Muslim Mathematicians,* 18.
[34] Ali Abdullah al-Daffa', *Sumbangan Islam dalam Bidang Matematik,* 43.

transposition of subtracted terms to the other side of the equation.[35] For example:

$$x = 50 - 5x \text{ is reduced to } 6x = 50$$

Whereas *al-Muqābalah* is the process of bringing quantities of the same type to the same side of the equation. Another theory of al-Muqabalah is to refer to the cancellation of terms on opposite side of the equation.[36] For example:

$$50 + x^2 = 20 + 10x \text{ is reduced to } x^2 + 30 = 10x.$$ [37]

In the book *al-Jabr wa al-Muqābalah*, three words were mentioned repeatedly, they are *al-jadhr* or *al-judhūr* as its plural, which means root, *al-māl* or *al-amwāl*, means square, and *al-'adad* which means number. The algebra deals basically with the theory of linear and quadratic equations with a single unknown. al-Khwārizmī's method in solving linear and quadratic equations worked by first reducing the equation to one of the six standard forms of which *a*, *b* and *c* are positive integers as stated below:

i. Squares equal roots $(ax^2 = bx)$
ii. Squares equal number $(ax^2 = c)$
iii. Roots equal number $(bx = c)$
iv. Squares and roots equal numbers $(ax^2 + bx = c)$
v. Squares and number equal roots $(ax^2 + c = bx)$
vi. Roots and number equal squares $(bx + c = ax^2)$

As previously mentioned before, the branch of algebra has become very vast in the world of mathematics. However, the basic algebra principles used in the modern world today are exactly the same as the three issues

[35] C. B. Boyer, *A History of Mathematics*, 229.
[36] *Ibid.*, 229.
[37] Hairudin Harun, *Daripada Sains Yunani kepada Sains Islam* (Kuala Lumpur: Universiti Malaya, 2004), 231.

mentioned by al-Khwārizmī. It is very significant to denote that the principles introduced by al-Khwārizmī marked the beginnings of algebra, and has been expanded extensively through centuries.

Elementary Algebra by al-Khwārizmī

In the first part of the book, there are three major issues brought in by him. They are multiplication and division, addition and subtraction, as well as measurement. These issues have been practiced and applied up until today, in modern world what is known as elementary algebra.

For the purpose of introducing the mathematical problems, al-Khwārizmī's approach in solving them is by explaining in the form of sentences. The following excerpts are taken from the English version from Roshdi Rashed's book. For example, for the equation of squares plus roots is equal to a number, here it is as mentioned in the book:[38]

> "...Squares plus roots are equal to a number", as when you say: a square plus ten roots are equal to thirty-nine dirhams; namely, if you add to any square <a quantity> equal to ten of its roots, the total will be thirty-nine.
>
> Procedure: you halve the number of the roots which, in this problem, yields five; you nultiply it by itself; the result is twenty-five; you add it to thirty-nine; the result is sixty-four; you take the root, that is eight, from which you substract half of the number of the roots, which is five. The remainder is three, that is the root of the square you want, and the square is nine..."

Therefore the solution in modern notation is:

[38] Roshdi Rashed, *al-Khwarizmi: The Beginnings of Algebra*, 101.

$$x^2 + 10x = 39$$

$$x^2 + 10x + \left(\frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2 = 39$$

$$x^2 + 10x + 5^2 - 25 = 39$$

$$(x+5)^2 = 39 + 25$$

$$x + 5 = \sqrt{64}$$

$$x + 5 = \pm 8$$

$$x = 3, x = -13$$

We can see from the above notation that squares plus roots are equal to a number which forms a quadratic equation.

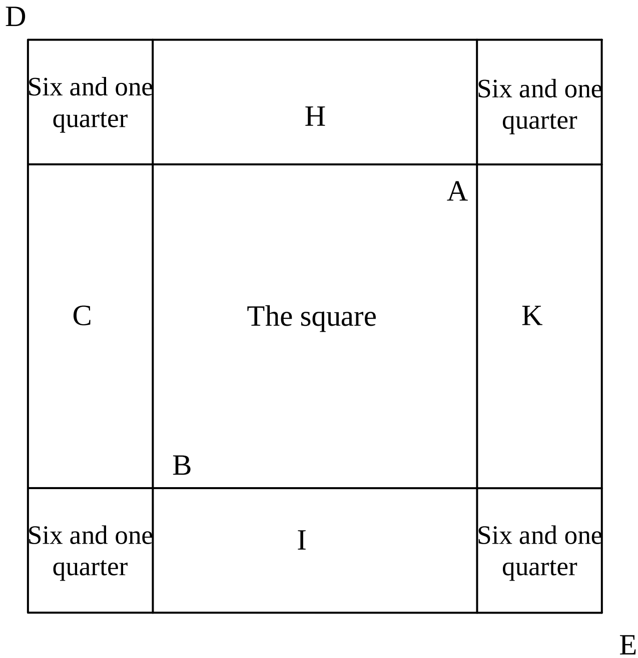
This is the kind of solution that has been accepted and used in general. However, al-Khwārizmī touches on solving the equations geometrically, in the form of diagrams. Here is an example:

> “... the square plus ten roots are equal to thirty-nine dirham”: The related figure of this is that of a square surface whose sides are unknown, which is the square you seek to know, and whose root you seek to know, that is the surface AB; each of its sides is its root, and if you multiply each if these sides by one number, then the numbers you get are the numbers if the roots. Each root is equal to the root of this surface.
>
> Thus, when it is said: With the square, there are ten of its roots; we take one-quarter of ten, which is two and one-half, and we make up each of these quarters with of the sides of the surface, thus, with the first surface, which is the surface AB, there will be four equal surfaces, whereby the length of each is equal to

the root of the surface AB, and whose width is two and one-half, Let these be the surfaces H,I,K,C. Thus, we have generated a surface of equal sides, that is also unknown, diminished in each of its four angles by two and a half <multiplied> by two and a half. This, in order to finish squaring the surface, it will be necessary to add two and a half by itself four times, and the sum of all this is twenty-five.

We have thus known that the first surface, which is the surface of the square, and the four surfaces surrounding it, which are ten of its roots, are thirty-nine in number, if we then add them twenty-five, which are the four squares, and which are on the angles of the surface AB, we then achieve the squaring if the largest surface, which is the surface DE. Now, we know that all this is sixty-four, and that one of its sides is its root, that is eight. Therefore, if we subtract from eight twice the quarter of ten, starting from the two extremes of the side of the largest surface, which is the surface DE that is five, the remainder of its side is three, which is equal to the side of the first surface – that is the surface AB – which is the root of this square.

We have divided the ten roots into two halves, and we have multiplied half of ten by itself, and added <the result> to the number, which is thirty-nine, in order that we complete the construction of the largest surface by what its four angles lacked. Since, if we multiply one-quarter of a number by itself, then by four, the product will be equal to that of multiplying its multiplication of half of the number of the roots by itself and then by four. Here is its figure:...”



Multiplication and Division

Besides from the diagram above, there are three other similar diagrams explaining similar mathematical problems on the issues of multiplication and division. After introducing six quadratic equations and their examples, al-Khwārizmī brought in issue on multiplication and division. Here is an example:[39]

> “...If you wish to multiply two roots of nine by three roots of four, extract the two roots of nine as I have described it to you, until you know of what square they are the root. Do the same for three roots of four, until you know of

[39] Roshdi Rashed, *al-Khwarizmi: The Beginnings of Algebra*, 137.

> what square they are the root. Then multiply the two squares by each other; the root of the product is <the resultant> of two roots of nine by the three roots of four..."

The solution in modern notation is:

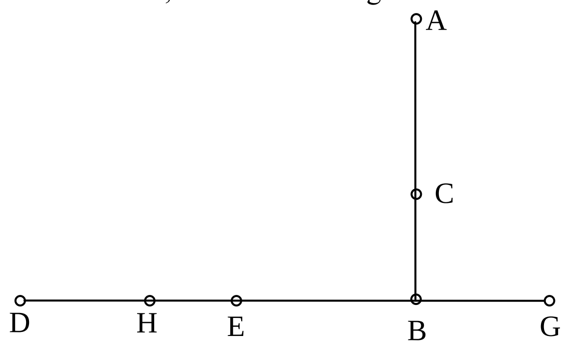
$$2\sqrt{9}\times 3\sqrt{4}$$
$$=2(3)\times 3(2)$$
$$=6\times 6$$
$$=36$$

In the following is al-Khwārizmī's explanation through the diagram:[40]

> "...With regard to the cause of "the root of two hundred minus ten, subtracted from twenty minus the root of two hundred", here is its figure:
>
> The line AB is the root of two hundred; from A to point C is ten, which is known. We produce a line from point B to point D, and we suppose that it is twenty. From point B to point E, we suppose <a line> equal to the line of the root of two hundred, which is equal to the line AB. It is clear that the line CB is what is left of the root of two hundred, once ten has been taken away; and the line ED is what is left of twenty, once the root of two hundred has been taken away. We want to subtract the line CB from the line ED. So we produce a line from point B to point G, which is equal to the line AC, which is ten. The whole GD will therefore be equal to the line GB plus the line BD. Now, it is clear that all that is thirty. From the line ED, we cut <a line> equal to the line CB; that

[40] *Ibid*., 137-139.

> is, the line EH. It is clear that the line HD is what is left of the whole of line GD, which is thirty; and it is clear that the line BE is the root of two hundred, and the line GB plus BC is also the root of two hundred. Since the line EH has become equal to the line CB, it will be clear that what has been subtracted from the line GD, which is thirty, is two roots of two hundred. But two roots of two hundred is the root of eight hundred. This is what we wanted to demonstrate, and here is its figure:..."



Addition and Subtraction

Similar thing happens in addition and subtraction. For example:[41]

> "...Know that the root of two hundred, minus ten, added to twenty minus the root of two hundred, is equal to ten..."

In modern notation:

$$\left(\sqrt{200}-10\right)+\left(20-\sqrt{200}\right)$$
$$=10$$

Another example:[42]

[41] Roshdi Rashed, *al-Khwarizmi: The Beginnings of Algebra*, 131.
[42] *Ibid.*.

> "...One hundred plus one square minus twenty roots, from which we subtract fifty plus ten roots minus two squares, is fifty dirhams plus three squares, minus thirty roots..."

In moden notation:

$$\left(100 + x^2 - 20x\right) - \left(50 + 10x - 2x^2\right)$$
$$= 100 - 50 + x^2 + 2x^2 - 20x - 10x$$
$$= 50 + 3x^2 - 30x$$

Various problems presented by al-Khwārizmī, combining multiplication, division, addition and subtraction in one problem, as the following is an example of equation in form of fractions:

> "...You have multiplied one-third of an amount plus one dirham by its quarter plus one dirham; the result is twenty.
>
> We thus infer that: you multiply one-third of a thing by one-quarter of a thing; the result is half of one-sixth of a square. You multiply one dirham by one-third of a thing, and the result is one-third of a thing; and one dirham by one dirham is one dirham. All that is half of one-sixth of a square, plus one-third of a thing, plus one-quarter of a thing, plus one dirham, and it is equal to twenty dirhams. Take away from twenty <the other> dirham, and there remain nineteen dirhams equal to half of one-sixth of a square, plus one-third of a thing, plus one-quarter of a thing. Complete your square; and for its completion, you multiply everything you have by twelve; what you have will be a square plus seven roots, which are equal to two hundred and twenty-eight dirhams. Divide the number of the roots into two halves: multiply the half by itself; and the product will be

twelve and one-quarter; add it to the number, which is two hundred and twenty eight; the result is two hundred and forty plus one-quarter. Take its root, which is fifteen and a half; subtract from it half the number of the roots, that is three and a half; the remainder is twelve, which is the amount. This problem has led you to one of the six procedures, which is "squares plus roots are equal to a number"... "

The solution in modern notation:

$$\left(\frac{1}{3}x+1\right)\left(\frac{1}{4}x+1\right)=20$$

$$\frac{1}{12}x^2+\frac{1}{3}x+\frac{1}{4}x=19$$

$$\frac{1}{12}x^2+\frac{4}{12}x+\frac{3}{12}x=19$$

$$x^2+7x=19\times 12$$

$$x^2+7x=228$$

$$x^2+7x+\left(\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2=228$$

$$\left(x+\frac{7}{2}\right)^2=228+\frac{49}{4}$$

$$x+\frac{7}{2}=\left(\sqrt{\frac{961}{4}}\right)$$

$$x=\frac{31}{2}-\frac{7}{2}$$

$$x=\frac{24}{2}$$

$$x=12$$

Another example:

> "...Likewise, if someone says: you divide ten into two parts, then you multiply one by the other; then you divide the product by the difference between the two parts – before you multiply them by each other -, so that the result is five and a quarter.
>
> We thus infer that: you take one thing from the ten, and the remainder is ten minus one thing. Multiply the one by the other, and the result is ten roots minus one square; this is the product of one of the two parts by the other. Then you divide this by the difference between the two parts; that is, ten minus two things; the quotient is five and a quarter. When you multiply five and a quarter by ten minus two things, you will have the amount, which was a product, which is ten things minus one square. Multiply five and a quarter by ten minus two things; the result is fifty-two dirhams and one-half minus ten roots and a half, equal to ten roots minus one square. Restore the fifty-two and a half by the ten roots and a half, and add them to the ten roots minus a square; then restore them by the square and add the square to the fifty-two dirhams plus one half; you will have twenty roots plus one half of a root equal to fifty-two dirhams plus one half plus one square. Reduce this, as we explained in the beginning of the book..."
>
> The solution in modern numbers:

$$\frac{x(10-x)}{10-2x} = 5+\frac{1}{4}$$

$$\frac{10x-x^2}{10-2x} = \frac{20+1}{4}$$

$$10x-x^2 = \frac{21}{4}(10-2x)$$

$$10x-x^2 = \frac{210-42x}{4}$$

$$4(10x-x^2) = 210-42x$$

$$40x-4x^2 - 210-42x$$

$$-4x^2 + 40x+42x = 210$$

$$-4x^2 + 82x = 210$$

$$4x^2 - 82x = -210$$

$$x^2 - 20\frac{1}{2}x = -52\frac{1}{2}$$

$$x^2 - 20\frac{1}{2}x + \left(-\frac{41}{4}\right)^2 - \left(-\frac{41}{4}\right)^2 = -52\frac{1}{2}$$

$$\left(x-\frac{41}{4}\right)^2 = -52\frac{1}{2} + \left(-\frac{41}{4}\right)^2$$

$$\left(x-\frac{41}{4}\right)^2 = \frac{-840+1681}{16}$$

$$x-\frac{41}{4} = \sqrt{\frac{841}{16}}$$

$$x = -\frac{29}{4}+\frac{41}{4}$$

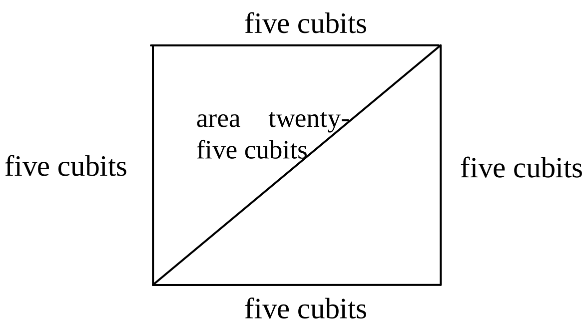
$$x = \frac{12}{4}$$

$$x = 3$$

Measurement

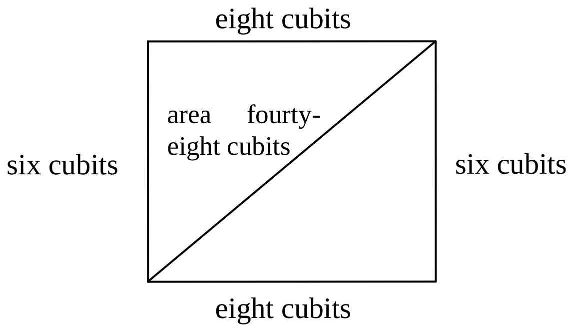
The last issue raised by al-Khwārizmī in the first part of his book is a chapter on measurement. For example:[43]

> "...For those among these quadrilaterals which have equal sides and right angles, or unequal sides and right angles, you multiply their length by their width in order to get their area. What you get is their area.
>
> Example: each side of a square piece of land is five qubits. Here is its figure:..."



> "...The second: A rectangular piece of land, whose lengths are eight cubits and eight cubits, and whose widths are six cubits and six cubits. You multiply by eight cubits by six cubits in order to get its area; the result is forty-eight cubits. This is its area.
>
> Here is its figure:..."

[43] Roshdi Rashed, *al-Khwarizmi: The Beginnings of Algebra,* p. 211 & 213.



Conclusion

If we look at the treatises written by al-Khwārizmī, we can conclude that al-Khwārizmī devoted his life not only in erudition of mathematics, but as well as in astronomy, geography, and other subjects in hope of seeking Allāh’s blessing. The idea of algebra presented by al-Khwārizmī may be perceived as easy and straightforward if we look from modern perspective. This is simply because we were taught in the context which has already been proven. However, if we look from the historical point of view, his ideas and contributions were very significant and his ability to discover those ideas was remarkable.

Debates on algebra in the book of *al-Jabr wa al-Muqābalah* were extensive and profound. There are a lot more that can be derived as a result of his theories on algebra. From the researcher’s point of view, the book does not revolve only around mathematics, but rather on other knowledge that can be obtained and developed. *Mu’āmalat*, *usūl fiqh* and Islamic jurisprudence are amongst the disciplines which are related to mathematics.

Indeed, a book as grand as this one should not be forgotten and should not be kept outdated. We may not realise that we are able to do the things that we do in our daily lives as a result of the hard works of ancient Islamic scholars. Such legacies are too valuable for the Muslims and it would be an absolute disgrace if we take it for granted. Instead, literatures from other renowned Islamic scholars should be recognised and developed as well.

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